Césaro Arrays I

Richard Stone

April 11, 2026

Abstract

To adapt H.L. Mencken's famous observation, every red-blooded mathematician at some time, when faced with making sense of a function defined as an infinite sum of more familiar functions, must have felt the temptation to expand all these familiar functions as power series, reverse the order of summation, and sum the series giving the coefficients at each degree to obtain a power series for the sum! Unfortunately, the series at each degree are usually divergent and the constraints of mathematical propriety advise that we therefore avert our eyes and abandon the effort. And even where a reckless disregard for mathematical manners permits the application of divergent series methods to make sense of these coefficient sums, a naive application of the above program still leads to the resulting summation function being erroneous. The purpose of this paper is to introduce the concept of Césaro arrays. These give an explicit framework, within the context of generalised geometric Césaro convergence (as introduced in previous papers), under which the above program can in fact be salvaged - the reversal of order of summation can be made to work, the divergent series for coefficients at each degree can be summed, and the resulting complications which have traditionally doomed the naive program-implementation can be resolved. Césaro arrays thus provide a new tool for analysing functions of a complex variable via power series (be they Taylor series, Laurent series or asymptotic series - at 0, ∞ or elsewhere). This is the first in a set of three papers exploring such arrays.

1 Introduction

Many interesting functions of a complex variable arise as a sum of other functions, for example the sum of exponentials $f(z) := \sum_{j=1}^{\infty} e^{-jz}$ or the sum of normal distributions $H(z) := \sum_{j=-\infty}^{\infty} e^{-\pi j^2 z^2}$. This latter function plays a central role in Riemann's initial paper[1] on the distribution of prime numbers - and its remarkable properties likewise play a critical role in Hardy's proof that there are infinitely many non-trivial zeros of ζ on the critical line, a proof which utilises the extraordinary structure of the singularities of H on the boundary of its region of convergence.

[1] in which the Riemann hypothesis was first conjectured

In such cases it is always tempting to try to understand the behaviour of the sum function, for z near 0 or as $z \to \infty$, by considering corresponding power series (e.g. Taylor, Laurent or asymptotic) for each summand function in z, reversing the order of summation, and summing the coefficients at each order, z^n, to obtain a power series for the sum function.

Unfortunately, in a world constrained by classical convergence, this is almost always impossible, partly because the series for the coefficients at each order are generally divergent, but also because some required orders in z are apparently missing altogether.

In a generalised geometric Césaro framework, however, we can overcome these obstacles and it is the purpose of this series of papers, of which this is the first, to outline how. The generalised geometric Césaro framework referred to here is as outlined in the three introductory papers ([I], [II] and [III]), and we note that the geometric aspects of this underlying theory (in terms of placement of summands and the use of a geometric Césaro summation variable) remain crucial in this new world of Césaro arrays.

In this paper we develop the Césaro arrays approach fully and apply it to derive some known and some new results as demonstration calculations. Specifically, our initial example calculation is a re-derivation which could instead easily be performed directly. But some of the subsequent examples, while known, are not straightforward and are only understood currently by alternative, indirect means; and the final demonstration calculation is completely new, as far as we know.

In the second paper in this series, we then apply these Césaro array methods to the function $H(z) := \sum_{j=-\infty}^{\infty} e^{-\pi j^2 z^2}$. Having derived its asymptotic behaviour near $z = 0$ in this paper, we use Césaro arrays to further derive its remarkable boundary singularity structure, mentioned above. We then also use this boundary Césaro array analysis for $H(z)$ to derive a famous theorem from the area of finite exponential sums and to extend it to obtain a further countable collection of related Césaro exponential sum identities. In this approach, the Césaro array analysis furnishes something akin to a generating function for such identities; and this suggests a more general application of such Césaro array methods to other problems in the field of exponential sums.

In the third and final paper in the series, we then return to comment on the rigourisation of some of the aspects of this Césaro array approach. This is necessary since the emphasis in the first two papers will be on illustrating the practical utility of such Césaro array methods, rather than having the advance guard get bogged down too early in the muddy wastes of formal proof or the hedgerows and fixed-trenches of detailed machinery.

2 Césaro arrays - methodology and examples

Preliminaries: The critical result which underlies the Césaro array method was introduced in [I] and then formally proven, as theorem 2, in [III]; namely that where we add in powers j^n ($n \in \mathbb{Z}_{\geq 0}$) at integer points j on the positive real axis, and let $X = k + \alpha$ in the usual way, then we have the following strong Césaro asymptotic relationship for the p-sum function:

$$s_{\zeta,-n}(X) \;=\; \sum_{j=1}^{k} j^n \;\overset{C}{\simeq}\; \frac{X^{n+1}}{n+1} \;+\; \zeta(-n) \;+\; o(1) \quad . \tag{1}$$

The *strong* Césaro asymptotic relationship captured by the notation $\overset{C}{\simeq}$ means that if we write the p-sum function as an exact relationship

$$\sum_{j=1}^{k} j^n \;=\; \frac{X^{n+1}}{n+1} \;+\; \zeta(-n) \;-\; R_n(X) \tag{2}$$

for some remainder-function, $R_n(X)$, then $R_n(X) \overset{C}{\to} 0$ as $X \to \infty$ via a pure power of the Césaro operator P, and in fact we have seen that

$$P^{n+1}[R_n](X) \to 0 \quad \text{classically as} \quad X \to \infty\,. \tag{3}$$

Indeed in [III] we saw that this remainder function can be represented explicitly, using the terminology of formal function elements, as

$$R_n(X) \;=\; (X - \overset{\vee}{q})^n \;:=\; \sum_{j=0}^{\infty} (-1)^j \binom{n}{j} X^{n-j} \overset{\vee}{q}{}^j \;=\; \sum_{j=0}^{\infty} (-1)^j \binom{n}{j} X^{n-j} \overset{\vee}{q}_j(\alpha) \tag{4}$$

where the $\overset{\vee}{q}_j(\alpha)$ are rescaled versions of the (periodised) Bernoulli polynomials and the final sum thus becomes a finite sum from $j = 0$ to $j = n$. We shall make use of this in the third in this suite of papers on Césaro arrays, where we discuss how to make this methodology rigorous, but for now it suffices to leave things as stated in equations 1-3.

The key point is that $s_{\zeta,-n}(X)$ contains only one component, namely $\frac{X^{n+1}}{n+1}$, which we might call strongly divergent in the sense of being a non-trivial eigenfunction of P, with eigenvalue $\frac{1}{n+2}$. The residual component of the expression for $s_{\zeta,-n}(X)$ is only weakly divergent in the sense that it can be made convergent purely by repeated application ($(n+1)$ times) of the Césaro averaging operator.

Let us now use this to consider our first example developing Césaro arrays.

2.1 The case of $f(z) := \sum_{j=1}^{\infty} e^{-jz}$ - developing the method of Césaro arrays

In the case of $f(z) := \sum_{j=1}^{\infty} e^{-jz}$ we can in fact calculate trivially what the correct asymptotic expansion for $f(z)$ as $z \to 0^+$ should be. If we let $r = e^{-z}$ then for $Re(z) > 0$ we have $|r| < 1$ and so $f(z)$ is given by the classically convergent geometric series

$$f(z) = \sum_{j=1}^{\infty} r^j = \frac{r}{1-r} = \frac{1}{e^z - 1} \quad .$$

Elementary manipulations thus give us that, as $z \to 0^+$,

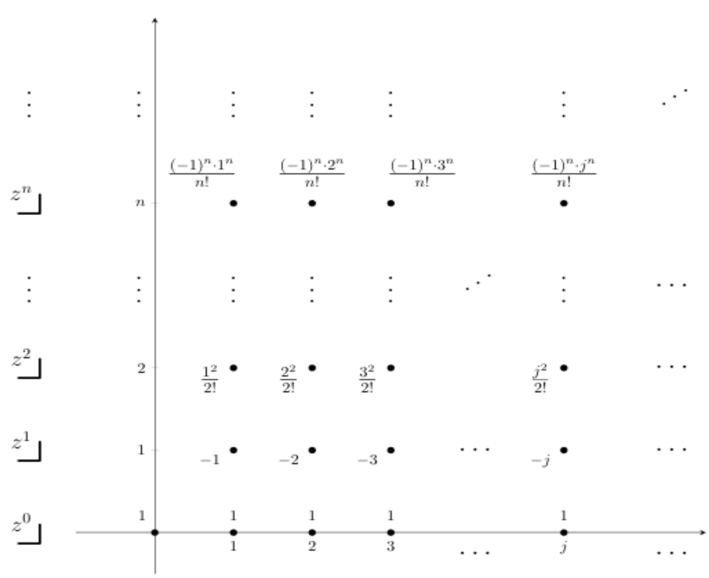
$$\begin{aligned} f(z) &= \frac{1}{z\{1+\frac{z}{2!}+\frac{z^2}{3!}+\ldots\}} \\ &= \frac{1}{z}\left\{1-\left[\frac{z}{2!}+\frac{z^2}{3!}+\ldots\right]+\left[\frac{z}{2!}+\frac{z^2}{3!}+\ldots\right]^2-\left[\frac{z}{2!}+\frac{z^2}{3!}+\ldots\right]^3+\ldots\right\} \\ &= \frac{1}{z}-\frac{1}{2}+\frac{1}{12}z-\frac{1}{720}z^3+\ldots \\ &= \frac{1}{z}+\sum_{j=0}^{\infty}(-1)^j\frac{\zeta(-j)}{j!}z^j\ . \end{aligned} \tag{5}$$

Note that the relationship of the coefficients in this expansion to the values of ζ at $0, -1, -2, \ldots$ is not apriori obvious in this approach. Rather, it just emerges as an observation from the computation.

Attempting an array approach: Now let us try to replicate result 5 using a naive program of expanding each summand e^{-jz} in its Taylor series near 0, reversing the order of summation and then summing independently at each degree, z^n. We place each summand, e^{-jz}, at the point j on the x-axis and if we expand e^{-jz} as

$$e^{-jz} = 1 - jz + \frac{j^2}{2!}z^2 - \frac{j^3}{3!}z^3 + \ldots$$

in the vertical direction by placing each term $(-1)^n\frac{j^n}{n!}z^n$ at (j,n) in the x-y plane, we obtain an array in which all terms involving z^n occur at the same "height" (i.e. at $(1,n)$, $(2,n)$, $\ldots$ in the 2-d plane). See Figure 1.



Reversing the order of summation means that rather than summing each column of terms vertically at $x = j$ to get e^{-jz} and then summing these horizontally, we instead try to sum each row in the array horizontally to get a coefficient of z^n, leaving a power series for $f(z)$ in powers of z near 0.

Problems: This leads to issues on two fronts trying to emulate equation 5:

(i) At height n the coefficient series for z^n is $\frac{(-1)^n}{n!}\sum_{j=1}^{\infty} j^n$ and this is classically divergent for all indexes $n \in \mathbb{Z}_{\geq 0}$. However this is easily resolved by adopting a generalised geometric Césaro framework. Then, as we saw in [I], this coefficient series converges in a generalised Césaro sense to $\frac{(-1)^n}{n!}\zeta(-n)$, which gives us the correct coefficient for z^n to match the power series for $f(z)$ in equation 5 for all $n \in \mathbb{Z}_{\geq 0}$. This looks very promising, and indeed has the virtue of showing naturally the connection of the coefficients in equation 5 to the non-positive integer values of ζ, which had formerly been somewhat surprising.

(ii) The second issue, however, seems harder to resolve. It is that equation 5 also has a term $\frac{1}{z}$ and this is entirely missing in the array approach as it stands. Nor is it obvious at first glance how it could possibly be recovered, since there are no terms at height -1 in our array to even sum up to get $\frac{1}{z}$.

A Césaro array: The way to resolve this difficulty and to get the full, correct power series expression 5 from our array approach, is to treat it as a "Césaro array" and conduct the Césaro treatment of the divergent horizontal coefficient sums at each height more carefully.

Specifically, letting $X = k + \alpha$ as usual, we have from results 2 and 3 that the p-sum function, $s_n(X)$, for the coefficient of z^n at height n is given by

$$s_n(X) = \frac{(-1)^n}{n!}\frac{X^{n+1}}{n+1} + \frac{(-1)^n}{n!}\zeta(-n) - \frac{(-1)^n}{n!}R_n[X] \tag{6}$$

where $P^{n+1}[R_n](X) \to 0$ classically as $X \to \infty$. Applying our generalised Césaro framework, we could now proceed in either of two ways.

One way is to treat the expression at each height as a single quantity and so to include a factor $\left(\frac{n+2}{n+1}\right)(P - \frac{1}{n+2})$ at each height n in order to annihilate the X^{n+1} divergence, along with the power P^{n+1} required to render the $R_n(X)$ term classically convergent to 0. This would lead us to apply a regular polynomial $q_n(P) = \left(\frac{n+2}{n+1}\right)(P - \frac{1}{n+2}) \cdot P^{n+1}$ at height n, which would leave us with just the term $\frac{(-1)^n}{n!}\zeta(-n)$ as the coefficient in the limit as $X \to \infty$.

But, as noted, while this would give the correct coefficient for each non-negative power of z, it would leave us without the required $\frac{1}{z}$ term in the overall power series. Furthermore, since ultimately we need to handle the series at all heights in our array simultaneously, it would require us to use not merely a regular polynomial $q(P)$ for the whole array, but rather some much more complex function of the operator P, and one containing the product

$\Pi_{n=0}^{\infty}\left(\frac{n+2}{n+1}\right)(P-\frac{1}{n+2})$.

Since the roots in this product have an accumulation point at 0, it is extremely unclear how we could make sense of any well-defined regular operator containing such a product[2]; and indeed it is not hard to see that finding such an operator is impossible since it would imply that any classically convergent entire function $g(X) := \sum_{j=0}^{\infty} a_j X^j$ would have to converge to limit a_0 as $X \to \infty$ and we know that this is not the case (consider e.g. $g(X) = e^{-X}$).

As such, while it is undoubtedly interesting to contemplate classes of regular operators, $q(P)$, more complex than simply regular polynomials, this is *not* the right approach to adopt in this instance to get our desired outcome from this array.

Instead we need to adopt the second, subtler way of proceeding, which is as follows. Rather than treating the entire expression for $s_n(X)$ as a single entity, we instead split it into two components.

The first of these combines the terms $\{\frac{(-1)^n}{n!}\zeta(-n) - \frac{(-1)^n}{n!}R_n[X]\}z^n$ and these continue to converge in a generalised Césaro sense to the correct degree-n term in 5 for each $n \in \mathbb{Z}_{\geq 0}$ - namely $\frac{(-1)^n}{n!}\zeta(-n)z^n$ - via a pure power of P, namelyP^{n+1}.

The second component consists simply of the single strongly-divergent term, $\frac{(-1)^n}{n!}\frac{X^{n+1}}{n+1}z^n$. These terms we do *not* now annihilate at each height. Rather, we leave them out of the final convergence step in each horizontal Césaro treatment, and instead re-combine just these pieces in their own vertical sum. In this case we get

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)!}X^{n+1}z^n \;=\; \frac{-1}{z}\cdot\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(n+1)!}X^{n+1}z^{n+1} \;=\; \frac{-1}{z}\{e^{-Xz}-1\}\;.$$

Since, for any given z with $Re(z) > 0$, we have $Re(Xz) \to \infty$ as $X \to \infty$, so this combined piece is no longer divergent, and in fact converges classically to $\frac{1}{z}$. It thus gives us precisely the additional term required, when combined with the power series terms from the first components above, in order to replicate equation 5.

This "Césaro array" approach thus successfully discovers the correct power series expression for $f(z)$ as $z \to 0^+$ in this example.

Comment: We have glided over one point in the above Césaro array treatment which warrants a little elaboration.

We no longer seek to Césaro-annihilate the strongly divergent piece $\frac{(-1)^n}{n!}\frac{X^{n+1}}{n+1}$ at each level n, and we thus avoid the difficulties in combining infinitely many factors $\left(\frac{n+2}{n+1}\right)(P-\frac{1}{n+2})$ discussed earlier. However the residual Césaro treatment of the first-component pieces in the n^{th} horizontal p-sum still requires the application of the pure power P^{n+1} - and since n is unbounded, this means there

[2]in particular, the overall coefficient in front of the factors in P will approach ∞ as we move through the heights n and include each new factor into the product

is no single finite power P^N which will suffice to handle the Césaro treatment of all the horizontal p-sums simultaneously.

Nevertheless, we would argue that application of the pure averaging operator P always acts to moderate divergences and render them "more convergent". As such it is at least plausible to suppose that doing "termwise" Césaro calculations at each height should be permissible as long as this only requires higher and higher pure powers of P (i.e. via *strongly* Césaro asymptotic relations at each level), rather than involving new factors of the form $\frac{1}{1-\lambda}(P-\lambda)$ at each stage.

In any case, this philosophy appears to work in this instance and so we shall take it as *defining* the general methodology of Césaro arrays as a tool for discovering power series expansions. In the third paper in this set, we will justify this heuristic reasoning in a broad class of cases, and comment further on this issue generally. For now, however, we remain focussed solely on utility of application - and so we conclude this discussion now with a clean summary of the precise steps which we define as constituting Césaro array analysis.

Summary of the Césaro array methodology: For a function which is given by a sum of other, building-block functions, we:

(a) Place each summand function suitably geometrically along the x-axis;

(b) Then expand each summand function as a power series vertically (in the y-direction) with corresponding powers for each summand placed at the same height; these power series could be convergent or asymptotic, and could be around 0 or ∞ or any other chosen point of interest;

(c) Then reverse the order of summation and use results 1 - 3 (or some similar generalised geometric Césaro result) to express the p-sum to $X = k + \alpha$ of each horizontal sum at height n (i.e. for order z^n) as a sum of two components. The first component consists of the generalised Césaro limit for the p-sum at that height, plus a residual piece which is Césaro-convergent to zero via some pure power of P. The second component consists of a finite sum of eigenfunctions and generalised eigenfunctions of P with eigenvalues not equal to 1 (i.e. some linear combination of powers, X^ρ, and power-log functions, $X^\rho(\ln(X))^m$, with $\rho \neq 0$);

(d) Then, working independently at each height n, apply the required power of P to make the first component converge to its Césaro limit, leaving a contribution of order z^n to the final sum at this height;

(e) On the other hand, leave second-component pieces at each height out of this termwise Césaro analysis. Collect these components from all the different levels separately and try to sum them vertically to obtain a combined function which is either classically convergent or else convergent in a generalised Césaro sense as $X \to \infty$ via a simple, regular polynomial $q(P)$ - and hence deduce an overall contribution from these components on letting $X \to \infty$ for a given z;

(f) Finally, combine the contributions from the second-component pieces in (e) with the first-component contributions at each level in (d), to obtain a final overall power series representation for the original sum function. Note that this power series representation may be convergent or it may be only asymptotic (with zero radius of convergence), and it applies for z approaching the point around which the expansions of the summand-functions were all taken.

Having laid out the methodology of Césaro arrays in this fashion, let us now test whether it continues to work in other interesting cases.

2.2 The case of $H(z) := \sum_{j=-\infty}^{\infty} e^{-\pi j^2 z^2}$ - testing the method of Césaro arrays

Let us consider the case of $H(z) := \sum_{j=-\infty}^{\infty} e^{-\pi j^2 z^2}$ and in particular its asymptotic behaviour as $z \to 0^+$. To simplify things and avoid bi-directional sums, we will in fact consider instead $\tilde{H}(z) := \frac{1}{2} + \sum_{j=1}^{\infty} e^{-\pi j^2 z^2}$, but since it is obvious that $\tilde{H}(z) = \frac{1}{2} H(z)$ so results for $\tilde{H}$ and H are interchangeable.

Unlike the case of $f(z) := \sum_{j=1}^{\infty} e^{-jz}$ considered in the previous section, there is no simple way of deriving the power series expansion for $\tilde{H}(z)$ analogous to our use of geometric progressions to understand $f(z)$, and elementary means seem unequal to the task. The way this asymptotic behaviour for $z \to 0^+$ has traditionally been derived in the literature is instead by indirect and rather advanced means.

Specifically, by using the fact that the function $e^{-\pi z^2}$ is essentially a fixed-point of the Fourier transform operator, and combining this with the Poisson summation formula, it is first shown[3] that $H(z)$ (and so also $\tilde{H}(z)$) satisfies the astonishingly simple functional equation that

$$H(z) = \frac{1}{z} H\left(\frac{1}{z}\right) \quad \text{or equivalently} \quad \tilde{H}(z) = \frac{1}{z} \tilde{H}\left(\frac{1}{z}\right) . \tag{7}$$

Now it is clear immediately from its definition that, as $z \to \infty$, we have

$$H(z) = 1 + \mathcal{S}_\infty(z) \tag{8}$$

where the notation $\mathcal{S}_\infty(z)$ denotes the set of Schwartzian functions near ∞, i.e. the set of functions which decay, and all of whose derivatives decay, faster than any power of z as $z \to \infty$; and equation 8 means that $H(z) - 1$ is Schwartzian as $z \to \infty$.

It therefore follows from the functional equation 7 that $H(z)$ has a surprisingly simple[4] asymptotic power series expansion for z near 0, namely that

$$H(z) = \frac{1}{z} + \mathcal{S}_0(z) \quad \text{as} \quad z \to 0^+ \tag{9}$$

[3] see e.g. [4, section 10.4]

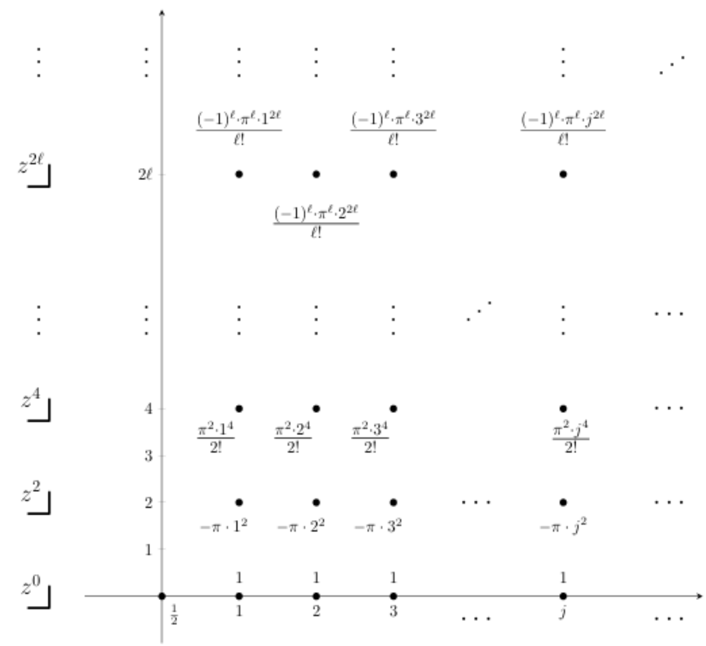
[4] Both the fact that H has such a simple singularity at 0 and the fact that it has no other terms (i.e. no power series terms of degree zero or higher, despite the smoothness of all its building-block functions near 0) is surprising, at least to this author!

or equivalently that

$$\tilde{H}(z) = \frac{1}{2}\frac{1}{z} + \mathcal{S}_0(z) \quad \text{as} \quad z \to 0^+ \,. \tag{10}$$

Césaro array derivation: Let us now instead consider the behaviour of $\tilde{H}(z)$ as $z \to 0^+$ using Césaro arrays and see if we can deduce this power series for $\tilde{H}(z)$ as $z \to 0^+$ directly, without needing to know its functional equation.

The Césaro array for $\tilde{H}(z)$ is as shown in Figure 2.



Using results 2-3 we see that, at height 0, the p-sum function is given by

$$s_0(X) = X + \frac{1}{2} + \zeta(0) - R_0(X) \tag{11}$$

while for height $n = 2l$, $l \in \mathbb{Z}_{>0}$, the p-sum function is given by

$$s_{2l}(X) = \frac{(-1)^l}{l!}\pi^l \left\{ \frac{X^{2l+1}}{2l+1} + \zeta(-2l) - R_{2l}(X) \right\} z^{2l} \,. \tag{12}$$

Since $\frac{1}{2} + \zeta(0) = 0$ and $\zeta(-2l) = 0$ for all $l \in \mathbb{Z}_{>0}$, while the $R_{2l}(X)$ all converge to 0 under the pure power P^{2l+1} for $l \in \mathbb{Z}_{\geq 0}$, it follows that our first-component pieces at each height contribute nothing to the final power series for $\tilde{H}(z)$ near 0, and this explains naturally the previously surprising fact that this power series has no terms of non-negative order. As for the second-component pieces, $\frac{(-1)^l}{l!}\pi^l \frac{X^{2l+1}}{2l+1} z^{2l}$, these combine vertically to give us the sum

$$\frac{1}{z}\sum_{l=0}^{\infty} \frac{(-1)^l}{l!}\pi^l \frac{(Xz)^{2l+1}}{2l+1} \;=\; \frac{1}{z}\int_0^{Xz} e^{-\pi u^2}\, du \,. \tag{13}$$

Since for any given $z \in \mathbb{R}_{>0}$ we have that $Xz \to \infty$ as $X \to \infty$, while also $\int_0^\infty e^{-\pi u^2}\, du = \frac{1}{2}$, it follows that this sum converges classically to $\frac{1}{2}\frac{1}{z}$ as $X \to \infty$.

Combining our contributions from these two components, we thus recover exactly the correct power series expansion, equation 10, for $\tilde{H}(z)$ as $z \to 0^+$ using our Césaro array methodology.

The method of Césaro arrays is thus again successful - it allows us to discover this power series expansion giving the asymptotic behaviour of $\tilde{H}$ near 0 by simple direct means, without having first to deduce the functional equation 7 or apply other advanced methods. Before leaving this example, we note a few additional points.

Observations: **(a)** Note that this example illustrates a key point not obvious from our first example. This is that the method of Césaro arrays can only ever tie down a function's asymptotic behaviour near a point (be that 0, ∞ or somewhere in between) modulo Schwartzian functions near that point. This is because the Césaro array approach calculates a power series for the sum function in a neighbourhood of that point, and Schwartzian functions consist of precisely those functions whose decay near the chosen point is too rapid to be captured by a power series. Why exactly the Césaro array method captured an *exact* power series for the sum function in the case of $f(z) := \sum_{j=1}^{\infty} e^{-jz}$, while only capturing a power series for $\tilde{H}(z)$ modulo $\mathcal{S}_0(z)$, is not a-priori clear. However, we will return to explain this in our final paper in this set.

(b) In this example, consider the Taylor series of each summand $e^{-\pi j^2 z^2} = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \pi^l j^{2l} z^{2l}$. In the third of our introductory papers (III) on generalised geometric Césaro methods, we introduced the concept of a coefficient "fabric" parametrised by a complex variable s, under which the coefficients of z^n in a power series lie embedded at the relevant integer points corresponding to the indices $s = n$.

In this case, the coefficients of the Taylor series for $e^{-\pi j^2 z^2}$ can be thought of as embedded in the coefficient fabric given by $\frac{\cos(\frac{\pi s}{2})}{\Gamma(\frac{s}{2}+1)} \pi^{\frac{s}{2}} j^s$ and we can think of the Taylor series as being a complete power series extension to all powers z^n, $-\infty < n < \infty$, but with coefficients which are zero for all $s = n$ odd (because then $\cos(\frac{\pi s}{2}) = 0$), and also for all $s = n$ a negative even integer (since then $\Gamma(\frac{s}{2} + 1)$ is singular).

Thought of this way, we can heuristically derive the overall $\frac{1}{2}\frac{1}{z}$ term in the power series for $\tilde{H}(z)$ via limiting calculations of the sort we undertook repeatedly in (III), rather than by sequestering and then vertically summing the strongly divergent eigenfunction divergences at each level in our array as per our stated Césaro array methodology given earlier.

Specifically, while the coefficient $\frac{\cos(\frac{\pi s}{2})}{\Gamma(\frac{s}{2}+1)} \pi^{\frac{s}{2}}$ is zero at $s = n = -1$, the horizontal sum at height -1 involves $\sum_{j=1}^{\infty} \frac{1}{j}$. This is Césaro-singular, since its p-sum involves a log-divergence, and this singularity offsets the zero coefficient.

Being more precise, we would evaluate the coefficient of z^{-1} at height -1 as

$$\begin{aligned}
\lim_{\epsilon\to 0}\frac{\cos(\frac{\pi(-1+\epsilon)}{2})}{\Gamma(\frac{(-1+\epsilon)}{2}+1)}\pi^{\frac{(-1+\epsilon)}{2}}\cdot\sum_{j=1}^{\infty}j^{-1+\epsilon} &= \lim_{\epsilon\to 0}\frac{\cos(\frac{-\pi}{2}+\frac{\pi\epsilon}{2})}{\Gamma(\frac{1}{2})}\pi^{\frac{-1}{2}}\cdot\zeta(1-\epsilon) \\
&= \frac{1}{\pi}\lim_{\epsilon\to 0}\sin(\frac{\pi\epsilon}{2})\cdot\left(-\frac{1}{\epsilon}+\gamma+O(\epsilon)\right) \\
&= -\frac{1}{2}
\end{aligned}$$

so that, up to a sign issue, we would get the correct contribution of $\frac{1}{2}\frac{1}{z}$ to the power series for $\tilde{H}(z)$ near 0.

This sign error can in fact be resolved (by utilising what physicists would call a "gauge-freedom" in our fabric definition) and this thus represents an intriguing alternative approach to deriving power series for sum-functions from arrays. As also discussed in [III], this use of fabrics and extension of power series "to the left" or "to the right" beyond their usual index-range is something to which we will return at length in a future set of papers on "Taylor series to the left".

For now, however, note that it is *not* part of Césaro array methodology as we have defined it above for this series of papers, and it would leave open the question of rigourisation even more profoundly than our stated methodology does, since rendering the horizontal series at each height Césaro convergent would entail including also the problematic collection of factors $(\frac{n+2}{n+1})(P-\frac{1}{n+2})$ with an accumulation point, which we discussed above.

(c) If we consider a slight variation $\tilde{H}_m(z) := \frac{1}{2}+\sum_{j=1}^{\infty}e^{-\pi j^2 z^{2m}}$, then it is trivial to see (either by change of variables or a repetition of our Césaro array analysis) that we still have $\tilde{H}_m(z)=\frac{1}{2}+\mathcal{S}_\infty(z)$ as $z\to\infty$, but now $\tilde{H}_m(z)=\frac{1}{2}\frac{1}{z^m}+\mathcal{S}_0(z)$ as $z\to 0^+$. The functions $\tilde{H}_m(z)$, $m\in\mathbb{Z}_{>0}$ - along with their products with pure powers, z^ρ, and together with other possible variants like $\tilde{H}_{m,k}(z) := \frac{1}{2}+\sum_{j=1}^{\infty}e^{-\pi j^{2k}z^{2m}}$ - thus give a useful tool-chest of functions which might be used to represent a given function (up to $\mathcal{S}_0$ and $\mathcal{S}_\infty$), based on a knowledge of its asymptotic power-series behaviour near $z=0$ and $z=\infty$.

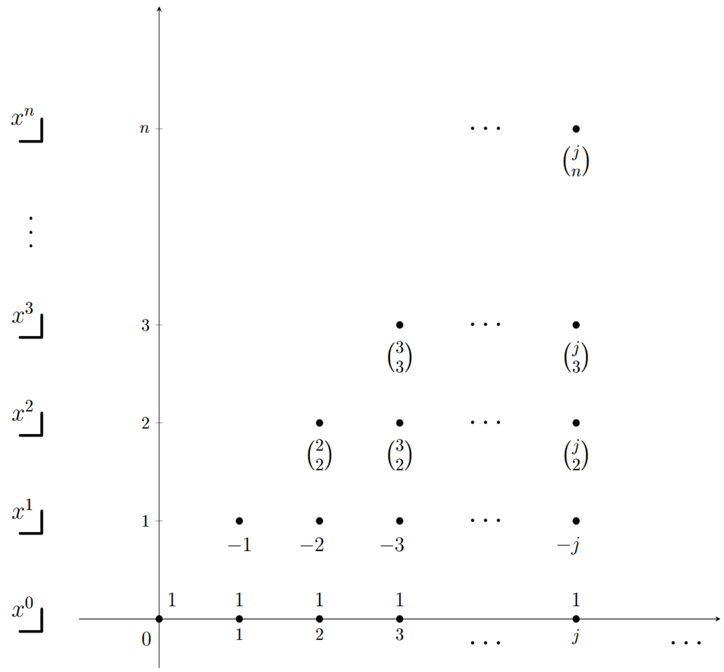
2.3 The case of $f(x) := \frac{1}{x}$ - an interesting aside re Césaro arrays

Before we turn to our last (and first wholly new) example showing the utility of the method of Césaro arrays, we turn briefly to consider an aside. As in our first example, this again involves a function which does not need Césaro array analysis, but which is thus ideally suited as a demonstration case to illustrate an interesting point about the Césaro array framework.

Consider the function $f(x) := \frac{1}{x}$. Obviously its asymptotic behaviour near 0 or ∞ does not need discovery! Nonetheless, it is interesting to write

$$\frac{1}{x}=\frac{1}{1-(1-x)}=1+(1-x)+(1-x)^2+\ldots=R_{+,0}[(1-x)^{\tilde{s}}](0) \qquad (14)$$

for x near 0 and consider the resulting Césaro array - see Figure 3.



Writing $X = k + \alpha$ as usual and letting $s_n(X)$ be the p-sum function at height n (i.e. at degree x^n), we see that at height 0 the p-sum function is given by $s_0(X) = k + 1 = \binom{k+1}{1}$; at height 1 it is $s_1(X) = -\{\sum_{j=1}^{k} \binom{j}{1}\}x = -\binom{k+1}{2}x$; at height 2 it is $s_2(X) = \{\sum_{j=1}^{k} \binom{j}{2}\}x^2 = \binom{k+1}{3}x^2$; and in general at height n it is

$$s_n(X) = \left\{ \sum_{j=1}^{k} (-1)^n \binom{j}{n} \right\} x^n = (-1)^n \binom{k+1}{n+1} x^n \; . \tag{15}$$

Attempting a continuous Césaro array treatment: If we try to apply our continuous Césaro array framework as before then, recalling from [I] that $Clim_{X\to\infty} k^l = \frac{(-1)^l}{l+1}$, we see that our first-component pieces give rise to an expansion of the form $\frac{1}{2} + \frac{1}{12}x + \frac{1}{24}x^2 + \frac{19}{720}x^3 + \ldots$. Thus the contribution from our second-component divergent pieces, on being combined and summed vertically, will need not only to supply our missing $\frac{1}{x}$ term, but also to cancel off all of these non-negative powers of x. It is not obvious how this might transpire if we proceed as in our first two examples, since the second-component pieces at heights $0, 1, 2, 3 \ldots$ are X, $-\frac{1}{2}X^2 x$, $\{\frac{1}{6}X^3 - \frac{1}{4}X^2\}x^2$, $-\{\frac{1}{24}X^4 - \frac{1}{6}X^3 + \frac{1}{6}X^2\}x^3$ and so forth, and it is not at all clear how these combine in the required way.

There is in fact a way to make sense of this vertical sum and so resolve the continuous Césaro array calculation in the way just outlined. However, like the recent digression into coefficient fabrics, it takes us in a new direction, involving a slightly different approach to the second-component calculations in our continuous Césaro array methodology. As such, we will defer this approach for just a moment - to the end of this section - so as to first outline an easier approach.

This easier way to resolve this example, while staying within our stated Césaro array methodology, is to recall that the Césaro framework includes not just a continuous approach for p-sum functions, but also a *discrete* approach for p-sum *sequences*, and it is this discrete approach which is much better adapted to the present problem.

A discrete Césaro array treatment: As outlined in [I], if we let P_D be the discrete Césaro operator on sequences $\{a_j\}_{j=-1}^{\infty}$ given by $P_D[\{a\}]_k = \frac{1}{k+2}\sum_{j=-1}^{k} a_j$ then our p-sum sequences $s_k = (-1)^n\binom{k+1}{n+1}x^n$ are in fact exact eigensequences of P_D (with eigenvalue $\frac{1}{n+2}$) for all $n \in \mathbb{Z}_{\geq 0}$.[5]

Thus, if we instead work in the discrete Césaro context for this Césaro array, we immediately get zero as the limit at each horizontal level from first-component pieces (since these pieces are all identically zero!). This leaves only the direct vertical re-combination of the divergent second-component eigensequences, which gives

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^n\binom{k+1}{n+1}x^n &= -\frac{1}{x}\sum_{n=1}^{\infty}(-1)^n\binom{k+1}{n}x^n \\ &= -\frac{1}{x}\left\{(1-x)^{k+1}-1\right\} \overset{C}{\to} \frac{1}{x} \qquad (16)\end{aligned}$$

since for $0 < x < 1$, $(1-x)^{k+1} \to 0$ classically as $k \to \infty$.

We thus do recover $\frac{1}{x}$ from the Césaro array approach in this instance, but we see that in this case the computation is vastly simplified by moving from the continuous to the discrete Césaro setting.

Fixing the continuous Césaro array treatment: To conclude this demonstration example, let us briefly return and outline how the continuous Césaro array calculation could have been made to work. Recall the result noted earlier from [I] (proven in [III]) that in the continuous Césaro framework we have

$$Clim_{X\to\infty}\, k^n = \frac{(-1)^n}{n+1} \quad \text{for all} \quad n \in \mathbb{Z}_{\geq 0} \quad , \qquad (17)$$

which can be re-expressed as

$$Clim_{X\to\infty}\, k^n = \int_{-1}^{0} k^n\, dk \quad \text{for all} \quad n \in \mathbb{Z}_{\geq 0}. \qquad (18)$$

This led to the first-component pieces giving a contribution $\frac{1}{2} + \frac{1}{12}x + \frac{1}{24}x^2 + \frac{19}{720}x^3 + \ldots$.

---

[5]The minor tweak to index our sequences starting at $j = -1$ is a trivial adjustment in order to arrange that sequences $\binom{k+1}{l}$ are our exact eigensequences of P_D, as adapted to this problem, rather than $\binom{k}{l}$ or $\binom{k-1}{l}$ etc.

Rather than express the second component-pieces in terms of X as we began doing before, however, let us instead therefore take the second-component pieces as $(-1)^n \left\{ \binom{k+1}{n+1} - \int_{-1}^{0} \binom{k+1}{n+1} \, dk \right\} x^n$ at each height n. Performing the vertical sum of these pieces as before, we thus obtain their contribution as

$$\begin{aligned} & -\frac{1}{x} \left\{ (1-x)^{k+1} - 1 \right\} + \frac{1}{x} \int_{-1}^{0} \left\{ (1-x)^{k+1} - 1 \right\} \, dk \\ = \; & \frac{1}{x} + \frac{1}{x} \left\{ \left[\frac{(1-x)^{k+1}}{\ln(1-x)} \right]_{-1}^{0} - 1 \right\} + o(1) \\ = \; & \frac{1}{x} + \frac{1}{x} \left\{ \frac{(1-x)}{\ln(1-x)} - \frac{1}{\ln(1-x)} - 1 \right\} + o(1) \\ \to \; & -\frac{1}{\ln(1-x)} \quad \text{as} \quad k \to \infty \, . \end{aligned}$$

An elementary expansion using that $\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \ldots$ then shows that $-\frac{1}{\ln(1-x)} = \frac{1}{x} - \frac{1}{2} - \frac{1}{12}x - \frac{1}{24}x^2 - \frac{19}{720}x^3 - \ldots$. Thus, when combined with the contribution from the first-component pieces, we get cancellation of all the non-negative powers of x and are left with just $\frac{1}{x}$, as desired.

We have thus shown that the continuous version of the Césaro array approach does work even here, albeit it is more convoluted than the corresponding discrete version. The above calculation has still been worth expounding, however, as an example showing that it is sometimes better to leave the second-component pieces at each height in terms of the discrete variable k rather than in terms of X as in the previous examples, and then to invoke the results 17 and 18 from papers [I] and [III].

2.4 A final - and new - example

There are many further instances where Césaro arrays are useful in deducing the asymptotic behaviour of sum functions near points of interest and where alternative methods are not readily apparent. In particular, in our next paper we will use them to deduce much of the singularity structure of $H(z)$ on the boundary of its domain of convergence, while in a future paper we shall consider them in the context of understanding the behaviour of the argument of the Riemann zeta function, $S(T)$, on the critical line $s = \frac{1}{2} + it$. For now, however, we conclude this paper with just one further example - the first completely new one considered so far. It illustrates some further interesting aspects of the methodology, but requires a little background.

Background: In [4, section 10.5 (pages 222 and 223)], as part of his discussion of Ramanujan's formula (much more on that in a future set of papers), the author considers the function $f(x)$ defined formally by the power series

$$f(x) := \sum_{j=0}^{\infty} B_{j+1} x^j = \sum_{j=0}^{\infty} (-1)^j (j+1) \zeta(-j) x^j \tag{19}$$

and the associated integral $\int_0^\infty x^{-s}\left\{\sum_{j=0}^\infty (-1)^j (j+1)\zeta(-j)x^j\right\} dx$, which he shows should formally equal $(-s)!s!\zeta(1-s)$.

Now the power series defining $f(x)$ is divergent for all $x \neq 0$, so that this integral as written is not in fact well-defined and the claimed identity is only formal. But by various clever algebraic manipulations it is shown that it can be transformed and rendered meaningful as the following equivalent integral identity:

$$\int_0^\infty y^{-s}\left\{\ln(y) - \frac{\Gamma'(y+1)}{\Gamma(y+1)}\right\} dy = \frac{\pi}{\sin(\pi s)}\zeta(s) \tag{20}$$

which is classically well-defined and true for all $0 < Re(s) < 1$ (per [5, formula 2.9.2]).

At no point in the working, however, is a direct meaning derived for the original function $f(x)$ defined by the problematic power series $\sum_{j=0}^\infty (-1)^j (j+1)\zeta(-j)x^j$. Now, using alternative means we can in fact deduce a prospective form for $f(x)$ from this power series, namely

$$f(x) = -\frac{1}{x} + \sum_{j=1}^\infty \frac{1}{(jx+1)^2} \; . \tag{21}$$

The derivation of this prospective explicit form from the given asymptotic power series involves using the idea of extending power series "to the left" outside their normal index-range - an idea which we have touched on in [III] and also earlier in this paper, and which is part of a much broader set of ideas that we will return to in a future set of papers. We do not elaborate on it here, however.

Instead we simply note that the function $g(x) := -\frac{1}{x} + \sum_{j=1}^\infty \frac{1}{(jx+1)^2}$ is a well-defined function for all $x \in (0,\infty)$ and we devote the rest of this section to showing, using Césaro arrays, that $g(x)$ is a function with the desired power series expansion for x near 0, so that $g(x)$ is in fact the correct form for $f(x)$ as claimed.

Césaro array treatment for $g(x)$: For any given j, as $x \to 0^+$, we have the Taylor series expansion

$$\frac{1}{(jx+1)^2} = 1 - 2jx + 3j^2x^2 - 4j^3x^3 + \dots \; . \tag{22}$$

Thus, placing each of these summand series vertically at j, we get the Césaro array shown in Figure 4 on the next page for $\sum_{j=1}^\infty \frac{1}{(jx+1)^2}$.

At each height n (i.e. degree x^n) it follows directly from results 2-3 that the p-sum function $s_n(X)$ is given by

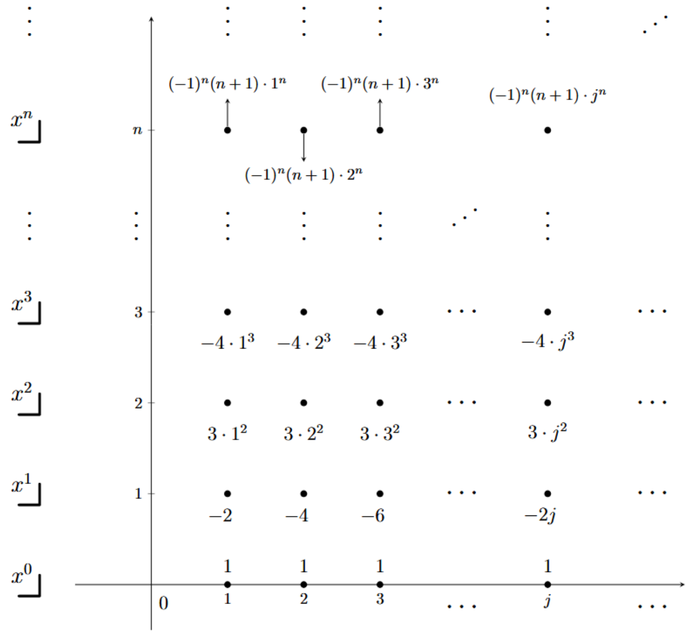
$$s_n(X) = (-1)^n(n+1)\left\{\frac{X^{n+1}}{n+1} + \zeta(-n) - R_n(X)\right\} x^n \quad . \tag{23}$$

Thus, taking Césaro limits by applying a sufficiently high power of P at each level, we get a contribution from first-component pieces consisting of

$$\zeta(0) - 2\zeta(-1)x + 3\zeta(-2)x^2 - 4\zeta(-3)x^3 + \dots \tag{24}$$

while the divergent second-component pieces (the X^{n+1} terms) combine vertically to give

$$X - X^2x + X^3x^2 - X^4x^3 + \dots\ . \tag{25}$$



Given any $x \in \mathbb{R}_{>0}$ then, taking X sufficiently small initially, this latter sum converges to $\frac{1}{x}\left(\frac{xX}{xX+1}\right)$; in this expression we may then let $X \to \infty$ and deduce that the second-component contribution becomes

$$\lim_{X\to\infty} \frac{1}{x}\left(\frac{xX}{xX+1}\right) \;=\; \frac{1}{x}\ . \tag{26}$$

Combining our contributions from 24 and 26 we thus get from our Césaro array analysis that

$$\sum_{j=1}^{\infty} \frac{1}{(jx+1)^2} \;=\; \frac{1}{x} + \sum_{j=0}^{\infty}(-1)^j(j+1)\zeta(-j)x^j \tag{27}$$

and thus the power series for $g(x)$ near 0 is given by

$$g(x) \;=\; \sum_{j=0}^{\infty}(-1)^j(j+1)\zeta(-j)x^j\ . \tag{28}$$

Since this coincides with the asymptotic power series we started with for $f(x)$ it follows that (up to $\mathcal{S}_0$), $g(x)$ does indeed give a correct functional form for

$f(x)$. Thus the function formally defined by Edwards via the asymptotic power series in equation 19, but never actually identified in [4], is in fact given by

$$f(x) \;=\; -\frac{1}{x} + \sum_{j=1}^{\infty} \frac{1}{(jx+1)^2} + \mathcal{S}_0(x)\;. \tag{29}$$

We see that this gives us another example showing the value[6] of our new Césaro array methodology in allowing us to understand the power series behaviour of sum functions from an understanding of the power series behaviour of the summands. It also nicely illustrates that this methodology continues to apply even where the resulting power series is merely an asymptotic series, with radius of convergence 0.

3 Acknowledgements

We thank Professor H.L. Mencken (pers. comm.) for many insightful comments, Professor T. Abby for his help in preparing this paper, and Christiana Stone for her help in preparing the figures.

References

[I] R. Stone, *Introduction to generalised Césaro convergence I*, 2026

[II] R. Stone, *Introduction to generalised Césaro convergence II*, 2026

[III] R. Stone, *Introduction to generalised Césaro convergence III*, 2026

[4] H.M. Edwards, *Riemann's Zeta Function*, Academic Press, 1974

[5] E.C. Titchmarsh, *The Theory of the Riemann Zeta-function*, Oxford Science Publications, Second Edition, Revised by D.R. Heath-Brown, Clarendon Press, Oxford, 1986

[6] together with our other machinery for guessing this functional form in the first place