Césaro Arrays II

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Abstract

$$\begin{gathered}Behold\ the\ lilies\ of\ the\ field,\ how\ they\ grow;\\ They\ toil\ not,\ neither\ do\ they\ spin:\\ And\ yet\ I\ tell\ you,\ that\ Solomon\ in\ all\ his\ glory\\ Was\ not\ arrayed\ like\ one\ of\ these.\end{gathered}$$

In this second in a series of three papers on Césaro arrays, we consider further the function $\tilde{G}(z) := \frac{1}{2} + \sum_{j=1}^{\infty} e^{-\pi j^2 z^2}$ which we analysed in [IV] and which plays a pivotal role in the theory of the Riemann zeta function. We apply the Césaro array results derived in [IV] for this function, together with the functional equation for $\tilde{G}(z)$, to deduce initial results regarding its extraordinary singularity structure on the boundary of its region of convergence. By directly applying an alternative 2-d Césaro array approach, we in turn use these initial boundary singularity results to deduce some famous identities regarding finite exponential sums. The method is new and provides an additional countable collection of Césaro sum identities in each case. The 2-d Césaro analysis undertaken covers the general setting of series with coefficients which are periodic, and we derive a number of results, as well as some further conjectures, for such series in general. These in turn, have further implications for the singularity structure of $\tilde{G}(z)$, which we enumerate.

1 Introduction

In [IV] we applied the methodology of Césaro arrays to the function $\tilde{G}(z) := \frac{1}{2} + \sum_{j=1}^{\infty} e^{-\pi j^2 z^2}$ to deduce its asymptotic behaviour near 0, namely that

$$\tilde{G}(z) = \frac{1}{2}\frac{1}{z} + \mathcal{S}_0(z) \quad \text{as} \quad z \to 0^+ \tag{1}$$

where $\mathcal{S}_0(z)$ is the set of Schwartzian functions near 0, i.e. the set of functions which decay, and all of whose derivatives decay, faster than any power of z as $z \to 0^+$.

This function (or equivalently the function $G(z) := \sum_{j=-\infty}^{\infty} e^{-\pi j^2 z^2} = 2\tilde{G}(z)$) plays a pivotal role in the theory of the Riemann zeta function. It appears in

Riemann's original paper[1] on the distribution of prime numbers, and it plays a critical role in Hardy's famous proof[2] that there are infinitely many non-trivial zeros of ζ on the critical line - a proof which depends upon the structure of the singularities of $\tilde{G}$ on the boundary of its region of convergence.

Using generalised geometric Césaro analysis, this paper fuses two avenues of further exploration related to $\tilde{G}(z)$. The first is an analysis of this structure of singularities and zeros of $\tilde{G}(z)$ on the boundary of its region of convergence.

The second relates to the evaluation of some famous finite exponential sums - in particular the evaluation, first carried out by Gauss, of $\sum_{j=0}^{p-1} e^{2\pi i \frac{j^2}{p}}$, and the evaluation of corresponding finite sums involving $e^{i\pi \frac{j^2}{p}}$ and more generally $e^{i\pi k \frac{j^2}{p}}$ where k and p are co-prime. The first of these cases (Gauss's sum) is usually quoted as requiring p to be an odd prime, but we show that the same result holds for arbitrary p odd without any assumption of primality.

The key to fusing these two areas of enquiry is to use two independent, but parallel, Césaro approaches in analysing $\tilde{G}(z)$ as z approaches suitably chosen points on the boundary of its region of classical convergence. On the one hand, we apply direct one-dimensional Césaro methods to the sum defining $\tilde{G}(z)$, in concert with using the simple functional equation for $\tilde{G}$. On the other, we convert the series defining $\tilde{G}(z)$ into a 2-d Césaro array with periodic coefficients and we develop further the theory for such arrays introduced in [IV] to obtain results in the general case of such periodic coefficients. From these we can then connect our finite exponential sums to the singular piece in the resulting power series expansion for $\tilde{G}$.

In more detail, in section 2 of this paper we begin to demonstrate the remarkable singularity structure of $\tilde{G}(z)$. We show that Schwartzian zeros and branch-point singularities are densely interleaved as we approach 0 along its convergence-boundary, which consists of the rays $Im(\ln(z)) = \pm\frac{\pi}{4}$.

In particular, we begin by taking a point $\nu_{2,p} := (\frac{-2i}{p})^{\frac{1}{2}} = \sqrt{\frac{2}{p}} e^{-\frac{i\pi}{4}}$ on this lower boundary, where p is an arbitrary odd positive integer, and we consider z approaching it at right angles from within the convergence region, so that $z^2 = -\frac{2i}{p} + \epsilon$ with $\epsilon \to 0^+$. By invoking the functional equation for $\tilde{G}$ and harnessing the dilation-invariance of Césaro summation (see [II]) we are then able to deduce that

$$\tilde{G}\left((-\frac{2i}{p} + \epsilon)^{\frac{1}{2}}\right) = \begin{cases} \frac{1}{2\sqrt{p}} \frac{1}{\sqrt{\epsilon}} + \mathcal{S}_0(\epsilon) & \text{if} \quad p \equiv 1\ (mod\ 4) \\ \\ \frac{i}{2\sqrt{p}} \frac{1}{\sqrt{\epsilon}} + \mathcal{S}_0(\epsilon) & \text{if} \quad p \equiv 3\ (mod\ 4)\,. \end{cases} \tag{2}$$

Thus $\tilde{G}(z)$ has a pure square-root branch-point *singularity* ($\tilde{G}(z) = \frac{C(p)}{\sqrt{z-\nu}} + \mathcal{S}_\nu$) at each such point $\nu = \nu_{2,p}$ on the lower boundary (a corresponding argument holds on the upper boundary).

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[1] in which the Riemann hypothesis was first conjectured

[2] see e.g. [5], section 11.1

We next consider $\nu = \nu_{1,p} := (\frac{-i}{p})^{\frac{1}{2}} = \sqrt{\frac{1}{p}}e^{-\frac{i\pi}{4}}$. We invoke the same argument and show that when p is even the point $\nu_{1,p}$ is likewise a branch-point singularity but that, by contrast, when p is odd $\nu_{1,p}$ is a Schwartzian *zero* since

$$\tilde{G}\left((-\frac{i}{p}+\epsilon)^{\frac{1}{2}}\right) = \mathcal{S}_0(\epsilon) \tag{3}$$

as $\epsilon \to 0^+$. Thus the points $\nu = \nu_{1,p}$ on the lower boundary are half branch-point singularities and half Schwartzian zeros. Since, as p gets larger and larger, the boundary points from these two sets ($\{\nu_{2,p}\}_{p\,odd}$ and $\{\nu_{1,p}\}_{p>0}$) accumulate arbitrarily closely as they both approach 0, this shows that $\tilde{G}$ has singularities and zeros interleaving arbitrarily closely as we approach 0 along the boundary of its domain of convergence.

The working to this point keeps the summations 1-dimensional; and the two key elements relied on in the derivations are the functional equation for $\tilde{G}$ and the dilation invariance of Césaro summation. Combining these two elements becomes considerably more difficult, however, when we try to move to a more general case such as considering $z^2 = -\frac{ik}{p} + \epsilon$ (i.e. considering z approaching $\nu_{k,p} := (\frac{-ik}{p})^{\frac{1}{2}}$) where k and p are arbitrary, but co-prime.

For this reason, and also in order to make contact with our second avenue of enquiry - the evaluation of finite exponential sums like $\sum_{j=0}^{p-1} e^{\frac{2\pi i j^2}{p}}$ - we change our approach in section 3 and instead expand our summations into true 2-d Césaro arrays. Specifically, for $z^2 = -\frac{2i}{p} + \epsilon$, we let $a_j = e^{\frac{2\pi i j^2}{p}}$, we place our summands $e^{-\pi j^2 z^2} = a_j e^{-\pi j^2 \epsilon}$ at the integer points j on the x-axis, and we expand each $e^{-\pi j^2 \epsilon}$ vertically in the Taylor series $e^{-\pi j^2 \epsilon} = \sum_{n=0}^{\infty} (-1)^n \pi^n \frac{j^{2n}}{n!} \epsilon^n$ in the y-direction.

The key to proceeding in a 2-d Césaro array calculation is then to note that the sequence $\{a_j\}$ is periodic and that, with minor adjustment it can be made into a stronger variety of periodic sequence, one which can be used to form the initial sequence in a discrete Césaro-adapted scale (see [III]).

Working in a *discrete* Césaro framework, we prove a number of results concerning such Césaro-adapted scales and these allow us to prove that the adjusted Césaro array (the array arising from the adjusted versions of the a_j) can be evaluated at each height using only pure powers of P_D, without any strongly-divergent "second-component" pieces (see [IV]) arising from non-trivial eigensequences of P_D. Since there are no second-component pieces to recombine, our "degree-wise" evaluation at each height thus provides a complete evaluation of the adjusted Césaro array.

We also show that the adjustment to the a_j required to make this all work is connected to the desired value of the finite exponential sum $\sum_{j=0}^{p-1} e^{\frac{2\pi i j^2}{p}}$. By combining our evaluation of the adjusted Césaro array with our results from section 2 we are thus able to read off the value of this exponential sum and hence confirm Gauss' famous evaluation (for p odd, not just p prime).

In fact Gauss' result corresponds solely to comparing the singular $\frac{1}{\sqrt{\epsilon}}$ terms in our two evaluations of $\tilde{G}(z)$. If we consider also the horizontal evaluations at each height (i.e. at order ϵ^0, ϵ^1, $\epsilon^2 \dots$) we get a further countable collection of Césaro sum identities closely connected to this first one and the 2-d Césaro array approach can be thought of as furnishing something like a generating function for additional such identities.

Having completed this analysis for the case of $z^2 = -\frac{2i}{p} + \epsilon$ we then replicate it for $z^2 = -\frac{i}{p} + \epsilon$, distinguishing between the cases of p even and p odd. This allows us to evaluate $\sum_{j=0}^{p-1} e^{\frac{i\pi j^2}{p}}$ when p is even, and $\sum_{j=0}^{2p-1} e^{\frac{i\pi j^2}{p}}$ when p is odd, as well as a countable collection of related identities in both cases.

We then extend to consider the general case of $z^2 = -\frac{ik}{p} + \epsilon$ (k, p co-prime). We show that, where our 1-d Césaro approach in section 2 had faltered, our 2-d Césaro array approach can often still make progress and allow us to draw conclusions about the behaviour of $\tilde{G}(z)$ as we approach $\nu_{k,p} = (\frac{-ik}{p})^{\frac{1}{2}}$.

For example, we are able to demonstrate that, whenever k and p are both odd, $\nu_{k,p}$ is a zero of $\tilde{G}(z)$, so that such zeros are densely packed all along the boundary of the region of convergence of $\tilde{G}$. And correspondingly we are able to give conditions for $\nu_{k,p}$ to be a square-root branch-point singularity which show that such singularities are generically common; indeed, we are able to show that there are certainly points arbitrarily far out along this boundary where such singularities and zeros accumulate densely.

We conclude section 3 with a brief note on discrete versus continuous Césaro frameworks and the need to be precise in regard to this choice of framework when splitting the first-component and second-component pieces in 2-d Césaro array analyses and examining them separately.

Finally, in section 4 we extend our analysis from section 3 to consider directly Césaro sums of the type which turned up there in our "degree-wise" calculations, namely those of the form $\sum_{j=1}^{\infty} a_j j^{2n}$ for $n \in \mathbb{Z}_{>0}$ where $\{a_j\}$ is periodic. We conjecture a general result giving relatively loose conditions under which such Césaro sums are all identically zero. Finally, we outline the steps involved in a proof of this conjecture and we perform them in detail to complete this proof in the cases of $n = 1$ and $n = 2$. Even these demonstration calculations point towards interesting new combinatorial Césaro results, including some which can be expressed in remarkably simple, compact form using formal symbols. We believe that extending from the cases of $n = 1$ and $n = 2$ to a proof of the conjecture in general should be both interesting and within reach for readers who possess even slightly greater combinatorial dexterity than the author and who are interested in accepting the challenge!

2 Singularities and zeros of $\tilde{G}(z)$ near 0 on the boundary of its region of convergence

It is easy to see that $\tilde{G}(z) = \frac{1}{2} + \sum_{j=1}^{\infty} e^{-\pi j^2 z^2}$ is classically convergent in the open wedge $-\frac{\pi}{4} < Im(\ln(z)) < \frac{\pi}{4}$, so that the boundary of its region of convergence consists of the two rays $Im(\ln(z)) = \pm\frac{\pi}{4}$. Let us examine how $\tilde{G}(z)$ behaves as we approach this boundary.

2.1 The behaviour of $\tilde{G}(z)$ near points $\nu_{2,p}$ on the boundary

As foreshadowed, take first a point $\nu_{2,p} = \sqrt{\frac{-2i}{p}}$ on the lower boundary[3], where $p \in \mathbb{Z}_{>0}$ is an arbitrary odd number, and consider the behaviour of $\tilde{G}(z)$ as z approaches this point at right-angles from within the region of convergence, i.e. let

$$z^2 = -\frac{2i}{p} + \epsilon \quad , \quad \epsilon \in \mathbb{R}_{>0} \tag{4}$$

and consider the behaviour of $\tilde{G}(z)$ as $\epsilon \to 0^+$. We have

$$\tilde{G}(z) = \frac{1}{2} + \sum_{j=1}^{\infty} e^{2\pi i \frac{j^2}{p}} e^{-\pi j^2 \epsilon} \tag{5}$$

but it is not immediately clear how to proceed directly in understanding $\tilde{G}(z)$ from this equation (we will return to this in section 3). Things are rendered tractable, however, if we recall that $\tilde{G}$ satisfies the simple functional equation

$$\tilde{G}(z) = \frac{1}{z}\tilde{G}(\frac{1}{z}) \ . \tag{6}$$

Based on equation 4, if we write $\frac{1}{z^2}$ as

$$\frac{1}{z^2} = \frac{ip}{2} + \hat{\epsilon} \tag{7}$$

where

$$\hat{\epsilon} = \frac{p^2}{4}\epsilon - i\frac{p^3}{8}\epsilon^2 - \frac{p^4}{16}\epsilon^3 + i\frac{p^5}{32}\epsilon^4 + \ldots \tag{8}$$

then

$$\begin{aligned} \frac{1}{z} &= \sqrt{\frac{ip}{2} + \hat{\epsilon}} = \frac{\sqrt{p}}{\sqrt{2}} e^{i\frac{\pi}{4}} \left(1 - i\frac{2}{p}\hat{\epsilon}\right)^{\frac{1}{2}} &(9) \\ &= \frac{\sqrt{p}}{\sqrt{2}} e^{i\frac{\pi}{4}} \left\{1 - \frac{i}{p}\hat{\epsilon} + \frac{1}{2p^2}\hat{\epsilon}^2 + \frac{i}{2p^3}\hat{\epsilon}^3 - \ldots\right\} \quad . &(10) \end{aligned}$$

Using the functional equation 6, our expression for $\tilde{G}(z)$ becomes

$$\begin{aligned} \tilde{G}(z) &= \frac{1}{z}\left\{\frac{1}{2} + \sum_{j=1}^{\infty} e^{-\pi\frac{j^2}{z^2}}\right\} \\ &= \frac{\sqrt{p}}{\sqrt{2}} e^{i\frac{\pi}{4}} \left(1 - i\frac{2}{p}\hat{\epsilon}\right)^{\frac{1}{2}} \cdot \left\{\frac{1}{2} + \sum_{j=1}^{\infty} e^{-i\frac{\pi}{2}pj^2} e^{-\pi j^2 \hat{\epsilon}}\right\} \ . \qquad (11) \end{aligned}$$

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[3] In this paper we will focus on the lower boundary, but all arguments can be trivially transposed to the upper boundary

This simplifies things greatly since we now have p in the numerator rather than the denominator of our exponent, so that

$$e^{-i\frac{\pi}{2}pj^2} = \begin{cases} 1 \quad , \quad \text{if } j \text{ even} \\ -i \quad , \quad \text{if } j \text{ odd and } p \equiv 1 \ (mod\ 4) \\ i \quad , \quad \text{if } j \text{ odd and } p \equiv 3 \ (mod\ 4)\,. \end{cases} \tag{12}$$

Thus $\frac{1}{2} + \sum_{j=1}^{\infty} e^{-i\frac{\pi}{2}pj^2} e^{-\pi j^2 \hat{\epsilon}}$ becomes the sum of two geometric Césaro sums:

$$\frac{1}{2} + \sum_{j=1}^{\infty} e^{-i\frac{\pi}{2}pj^2} e^{-\pi j^2 \hat{\epsilon}} = \begin{cases} A - iB \quad , \ p \equiv 1 \ (mod\ 4) \\ \\ A + iB \quad , \ p \equiv 3 \ (mod\ 4) \end{cases} \tag{13}$$

where A is the Césaro sum of the geometric picture shown in Figure 1:

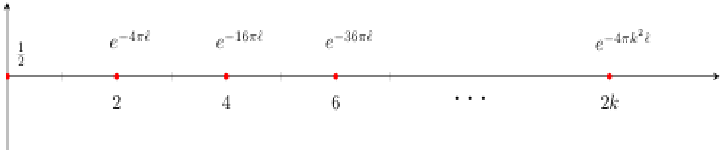


Figure 1

and B is the Césaro sum of the geometric picture shown in Figure 2:

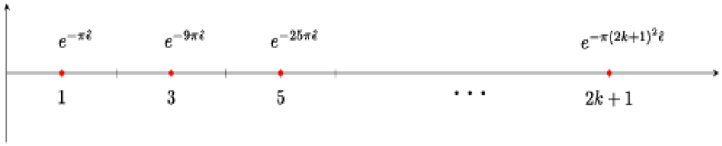


Figure 2

Now for u real, $\tilde{G}(u) = \frac{1}{2} + \sum_{j=1}^{\infty} e^{-\pi j^2 u^2}$ corresponds to the picture in Fig. 3:

Figure 3

and in [IV] we derived, using Césaro methods, that it satisfies

$$\tilde{G}(u) = \frac{1}{2}\frac{1}{u} + \mathcal{S}_0(u) \quad \text{as} \quad u \to 0^+ \,. \tag{14}$$

In [II] we also proved that geometric Césaro summation is dilation-invariant. It follows, on dilating so as to place its summands at $0, 1, 2, \ldots$ rather than at

$0, 2, 4, \ldots$ and on taking $u^2 = 4\hat{\epsilon}$, that the first of these Césaro pictures for A gives

$$A = \frac{1}{4}\frac{1}{\sqrt{\hat{\epsilon}}} + \mathcal{S}_0(\hat{\epsilon}) \quad \text{as} \quad \hat{\epsilon} \to 0^+ \,. \tag{15}$$

As for the Césaro picture for B, this satisfies $B = B' - A$ where B' is the Césaro sum of the following picture (Figure 4):

Figure 4

It follows, by equation 15 and by equation 14 applied to the picture for B', that we have

$$\begin{aligned} B &= \left\{\frac{1}{2}\frac{1}{\sqrt{\hat{\epsilon}}} + \mathcal{S}_0(\hat{\epsilon})\right\} - \left\{\frac{1}{4}\frac{1}{\sqrt{\hat{\epsilon}}} + \mathcal{S}_0(\hat{\epsilon})\right\} \\ &= \left\{\frac{1}{4}\frac{1}{\sqrt{\hat{\epsilon}}} + \mathcal{S}_0(\hat{\epsilon})\right\} \quad \text{as} \quad \hat{\epsilon} \to 0^+ \,. \end{aligned} \tag{16}$$

In equation 13 we thus have that

$$\frac{1}{2} + \sum_{j=1}^{\infty} e^{-i\frac{\pi}{2}pj^2} e^{-\pi j^2 \hat{\epsilon}} = \begin{cases} \sqrt{2}\, e^{-i\frac{\pi}{4}} \frac{1}{4}\frac{1}{\sqrt{\hat{\epsilon}}} + \mathcal{S}_0(\hat{\epsilon}) & ,\ p \equiv 1\ (mod\ 4) \\ \\ \sqrt{2}\, e^{i\frac{\pi}{4}} \frac{1}{4}\frac{1}{\sqrt{\hat{\epsilon}}} + \mathcal{S}_0(\hat{\epsilon}) & ,\ p \equiv 3\ (mod\ 4) \end{cases} \,. \tag{17}$$

and hence in equation 11 we have that

$$\tilde{G}(z) = \begin{cases} \frac{\sqrt{p}}{4}\, \frac{\left(1 - i\frac{2}{p}\hat{\epsilon}\right)^{\frac{1}{2}}}{\sqrt{\hat{\epsilon}}} + \mathcal{S}_0(\hat{\epsilon}) & ,\ p \equiv 1\ (mod\ 4) \\ \\ i\frac{\sqrt{p}}{4}\, \frac{\left(1 - i\frac{2}{p}\hat{\epsilon}\right)^{\frac{1}{2}}}{\sqrt{\hat{\epsilon}}} + \mathcal{S}_0(\hat{\epsilon}) & ,\ p \equiv 3\ (mod\ 4) \end{cases} \tag{18}$$

It remains only to convert back from an expression in $\hat{\epsilon}$ to an expression in ϵ. This can be done long-hand by replacing $\hat{\epsilon}$ with ϵ using equation 8 (and doing so is all good, clean fun!), but a much quicker simplification can be achieved by instead recalling that in equations 4 and 7, $\hat{\epsilon}$ is *defined* via the relationship

$$\frac{1}{\nu_{2,p}^2 + \epsilon} = \frac{1}{\nu_{2,p}^2} + \hat{\epsilon} \tag{19}$$

and thus

$$\hat{\epsilon} = \frac{1}{\nu_{2,p}^2 + \epsilon} - \frac{1}{\nu_{2,p}^2} \,. \tag{20}$$

It follows that

$$\begin{aligned}
\frac{\left(1-i\frac{2}{p}\hat{\epsilon}\right)^{\frac{1}{2}}}{\sqrt{\hat{\epsilon}}} &= \frac{\left(1+\nu_{2,p}^2\hat{\epsilon}\right)^{\frac{1}{2}}}{\sqrt{\hat{\epsilon}}} = \nu_{2,p}\cdot\frac{\left(\frac{1}{\nu_{2,p}^2}+\hat{\epsilon}\right)^{\frac{1}{2}}}{\sqrt{\hat{\epsilon}}} \\
&= \nu_{2,p}\cdot\frac{1}{\sqrt{\nu_{2,p}^2+\epsilon}}\cdot\frac{1}{\sqrt{\frac{1}{\nu_{2,p}^2+\epsilon}-\frac{1}{\nu_{2,p}^2}}} \\
&= \frac{\nu_{2,p}}{\sqrt{1-\frac{\nu_{2,p}^2+\epsilon}{\nu_{2,p}^2}}} = \frac{\nu_{2,p}}{\sqrt{\frac{-\epsilon}{\nu_{2,p}^2}}} \\
&= \frac{2}{p}\cdot\frac{1}{\sqrt{\epsilon}} \qquad (21)
\end{aligned}$$

with no other terms at orders $\epsilon^0, \epsilon^1, \epsilon^2, \ldots$! Thus in equation 18 we get finally the following expression for $\tilde{G}(z)$ as $z \to \nu_{2,p}$, i.e. when $z^2 = -\frac{2i}{p} + \epsilon$ and $\epsilon \to 0^+$:

$$\tilde{G}(z) = \begin{cases} \frac{1}{2\sqrt{p}}\,\frac{1}{\sqrt{\epsilon}} + \mathcal{S}_0(\epsilon) & , p \equiv 1\ (mod\ 4) \\ \\ \frac{i}{2\sqrt{p}}\,\frac{1}{\sqrt{\epsilon}} + \mathcal{S}_0(\epsilon) & , p \equiv 3\ (mod\ 4) \quad . \end{cases} \qquad (22)$$

The points $\nu_{2,p} = \sqrt{\frac{-2i}{p}}$ are thus singular points of $\tilde{G}(z)$ for any odd positive integer p. In fact in equation 4 we have that

$$z = \sqrt{\nu_{2,p}^2+\epsilon} = \nu_{2,p}\cdot\sqrt{1+\frac{\epsilon}{\nu_{2,p}^2}} = \nu_{2,p}\cdot\left\{1+\frac{1}{2}\frac{\epsilon}{\nu_{2,p}^2}+O(\epsilon^2)\right\}$$

so that $\epsilon \approx 2\nu_{2,p}(z-\nu_{2,p})$ and $\sqrt{\epsilon} \approx \sqrt{2\nu_{2,p}}\sqrt{(z-\nu_{2,p})}$. Thus, after elementary simplifications, we see that

$$\tilde{G}(z) = \begin{cases} C_{2,p}\cdot\frac{1}{\sqrt{(z-\nu_{2,p})}} + \mathcal{S}_0(z-\nu_{2,p}) & , p \equiv 1\ (mod\ 4) \\ \\ iC_{2,p}\cdot\frac{1}{\sqrt{(z-\nu_{2,p})}} + \mathcal{S}_0(z-\nu_{2,p}) & , p \equiv 3\ (mod\ 4) \end{cases} \qquad (23)$$

as z approaches $\nu_{2,p}$, where $C_{2,p} = \frac{e^{i\frac{\pi}{8}}}{2^{\frac{7}{4}}p^{\frac{1}{4}}}$. That is, each point $\nu_{2,p}$ is in fact a square-root branch-point singularity of $\tilde{G}$, with a coefficient at the branch-point which decays like $p^{-\frac{1}{4}}$ as p increases and $\nu_{2,p}$ approaches 0.

2.2 The behaviour of $\tilde{G}(z)$ near points $\nu_{1,p}$ on the boundary

Using the same methods as above, let us now consider instead the alternative set of points $\nu_{1,p} = \sqrt{\frac{-i}{p}}$ on the lower boundary, where $p \in \mathbb{Z}_{>0}$ is now arbitrary. We set

$$z^2 = -\frac{i}{p} + \epsilon \quad , \quad \epsilon \in \mathbb{R}_{>0} \qquad (24)$$

and write

$$\frac{1}{z^2} = ip + \hat{\epsilon} \tag{25}$$

so that $\hat{\epsilon} = p^2\epsilon + O(\epsilon^2)$, and

$$\frac{1}{z} = \sqrt{ip + \hat{\epsilon}}\ . \tag{26}$$

Invoking the functional equation for $\tilde{G}$ as before, we thus have

$$\begin{aligned} \tilde{G}(z) &= \frac{1}{z}\left\{\frac{1}{2} + \sum_{j=1}^{\infty} e^{-\pi\frac{j^2}{z^2}}\right\} \\ &= \sqrt{p}e^{i\frac{\pi}{4}}\left(1 - \frac{i}{p}\hat{\epsilon}\right)^{\frac{1}{2}} \cdot \left\{\frac{1}{2} + \sum_{j=1}^{\infty} e^{-i\pi p j^2} e^{-\pi j^2\hat{\epsilon}}\right\} \end{aligned} \tag{27}$$

and things are even simpler since when p is even, $e^{-i\pi pj^2} = 1$ for all j; while if p is odd then

$$e^{-i\pi pj^2} = \begin{cases} 1 & \text{if } j \text{ even} \\ \\ -1 & \text{if } j \text{ odd} \end{cases} \tag{28}$$

Case 1 [p even]: If p is even we thus get that $\frac{1}{2} + \sum_{j=1}^{\infty} e^{-i\pi pj^2} e^{-\pi j^2\hat{\epsilon}}$ is the Césaro sum of the geometric picture given earlier for B' in Figure 4, and thus, as previously calculated using equation 14, we have that

$$\frac{1}{2} + \sum_{j=1}^{\infty} e^{-i\pi pj^2} e^{-\pi j^2\hat{\epsilon}} = \frac{1}{2}\frac{1}{\sqrt{\hat{\epsilon}}} + \mathcal{S}_0(\hat{\epsilon})\ . \tag{29}$$

It follows, by an identical derivation to that given in deducing equation 21, that for p even we have

$$\begin{aligned} \tilde{G}(z) &= \sqrt{p}e^{i\frac{\pi}{4}}\left(1 - \frac{i}{p}\hat{\epsilon}\right)^{\frac{1}{2}} \cdot \left\{\frac{1}{2}\frac{1}{\sqrt{\hat{\epsilon}}}\right\} + \mathcal{S}_0(\hat{\epsilon}) \\ &= \frac{e^{i\frac{\pi}{4}}}{2\sqrt{p}}\frac{1}{\sqrt{\epsilon}} + \mathcal{S}_0(\epsilon) \quad \text{as} \quad \epsilon \to 0^+\ . \end{aligned} \tag{30}$$

For p even, the points $\nu_{1,p} = \sqrt{\frac{-i}{p}}$ are thus also all square-root branch-point singularities of $\tilde{G}(z)$ (i.e. $\tilde{G}(z) \approx \frac{C(p)}{\sqrt{(z-\nu_{1,p})}} + \mathcal{S}_0(z - \nu_{1,p})$) on its lower convergence boundary.

Case 2 [p odd]: When p is odd, however, a different situation emerges as z approaches $\nu_{1,p}$. In this case, by equation 28, we get that $\frac{1}{2}+\sum_{j=1}^{\infty} e^{-i\pi pj^2} e^{-\pi j^2\hat{\epsilon}} =$

$A - B$ where A and B are as given previously, namely the Césaro sums of the geometric pictures given earlier in Figures 1 and 2 respectively. It follows, by equations 15 and 16, that

$$\begin{aligned} \frac{1}{2} + \sum_{j=1}^{\infty} e^{-i\pi p j^2} e^{-\pi j^2 \hat{\epsilon}} &= \left\{ \frac{1}{4}\frac{1}{\sqrt{\hat{\epsilon}}} + \mathcal{S}_0(\hat{\epsilon}) \right\} - \left\{ \frac{1}{4}\frac{1}{\sqrt{\hat{\epsilon}}} + \mathcal{S}_0(\hat{\epsilon}) \right\} \\ &= \mathcal{S}_0(\hat{\epsilon}) \;=\; \mathcal{S}_0(\epsilon) \quad \text{as} \quad \epsilon \to 0^+ \, . \end{aligned} \tag{31}$$

Thus, in this case

$$\tilde{G}(z) \;=\; \mathcal{S}_0(\epsilon) \quad \text{as} \quad \epsilon \to 0^+ \tag{32}$$

and so, when p is odd, the points $\nu_{1,p} = \sqrt{\frac{-i}{p}}$ are not singular, but in fact are all Schwartzian zeros.

2.3 Initial conclusions

Combining the results from sections 2.1 and 2.2 we can already conclude that $\tilde{G}(z)$ has a remarkable singularity structure on the boundary of its region of convergence in any neighbourhood of 0. Square-root branch point singularities arising from points $\nu_{2,p}$ (p odd) and $\nu_{1,p}$ (p even) accumulate densely along this boundary as we approach 0 (there are infinitely many such singularities in any neighbourhood of 0), but they are discrete and are interspersed with a corresponding dense set of discrete Schwartzian zeros arising from points $\nu_{1,p}$ (p odd), of whom there are also infinitely many in any neighbourhood of 0.

This provides a first glimpse into what we (following Edwards) have referred to earlier as the extraordinary singularity structure of $\tilde{G}(z)$ along the whole boundary of its region of convergence; a structure which, as noted earlier, is crucial in Hardy's famous proof that there are infinitely many non-trivial zeros of ζ on the critical line $Re(s) = \frac{1}{2}$ (albeit that in fact, for Hardy's proof it is really only that $\nu_{1,1}$ is a Schwartzian zero which is relied upon - see [5], section 11.1).

Is the structure of singularities and zeros of $\tilde{G}(z)$ equally wild elsewhere on its boundary, i.e. in neighbourhoods of points $z_0 \neq 0$ on its convergence boundary? In particular, if we let $\nu_{k,p} = \sqrt{\frac{-ik}{p}}$ for arbitrary k, p co-prime, but with the factor k on the numerator now also unbounded, can we make similar deductions regarding whether such points are singularities or zeros of $\tilde{G}(z)$ as z approaches $\nu_{k,p}$ at right angles?

Well, if we try to replicate the arguments of sub-sections 2.1 and 2.2 then we immediately encounter difficulties when $k \geq 3$, since when $k \neq 1, 2$ the application of the functional equation for $\tilde{G}(z)$ leaves us with an expression $\frac{1}{2} + \sum_{j=1}^{\infty} e^{-i\pi \frac{p}{k} j^2} e^{-\pi j^2 \hat{\epsilon}}$, but the "coefficients" $e^{-i\pi \frac{p}{k} j^2}$ no longer form such a simple pattern. While still periodic, they are not obviously susceptible to re-arrangement using Césaro dilation in a way that reduces just to expressions involving the known Césaro asymptotic sum given in equation 14. As such, while it might be interesting to explore this approach further, for now we conclude that the 1-dimensional Césaro methods which we applied in sections 2.1 and 2.2 for $k = 2$ and $k = 1$ do not extend to arbitrary $k \geq 3$. Instead, to tackle this case, we change tack and move from 1-d Césaro methods reliant on the functional equation for $\tilde{G}(z)$, to methods based on 2-d Césaro arrays and a careful use of coefficient periodicity. This in turn allows us to make contact with the field of finite exponential sums and their evaluation.

3 Singularities and zeros of $\tilde{G}(z)$ elsewhere on its boundary - periodicity, 2-d Césaro arrays and the evaluation of finite exponential sums

A famous result from the field of exponential sums, which deals with the evaluation of finite sums of oscillatory exponentials, is the result due to Gauss that, for p an odd prime,

$$\sum_{j=0}^{p-1} e^{\frac{2\pi i}{p} j^2} = \begin{cases} \sqrt{p} & , p \equiv 1 \ (mod\ 4) \\ \\ i\sqrt{p} & , p \equiv 3 \ (mod\ 4) \end{cases} \tag{33}$$

Comparing with equation 22 it seems at least plausible that this result is connected with the coefficients of the singular term in the expansion of $\tilde{G}(z)$ as z approaches $\nu_{2,p} = \sqrt{\frac{-2i}{p}}$, i.e. where $z^2 = -\frac{2i}{p} + \epsilon$ and we let $\epsilon \to 0^+$. And this connection is strengthened when we recall that, at such z,

$$\tilde{G}(z) = \frac{1}{2} + \sum_{j=1}^{\infty} e^{\frac{2\pi i}{p} j^2} e^{-\pi j^2 \epsilon} \tag{34}$$

so that (modulo a caveat at $j = 0$) the terms in the finite exponential sum 33 appear as the periodic coefficients in the infinite sum for $\tilde{G}(z)$.

Together with our remarks in section 2.3, this observation motivates us to analyse $\tilde{G}(z)$ directly, by considering it as a sum whose "coefficients" $e^{\frac{2\pi i}{p} j^2}$ are periodic in j, and whose terms $e^{-\pi j^2 \epsilon}$ we then expand out in the "vertical direction" to form a 2-d Césaro array to which we can then apply the evaluation methodology we developed in [IV].

Specifically, widening our perspective temporarily, let us consider the general case of

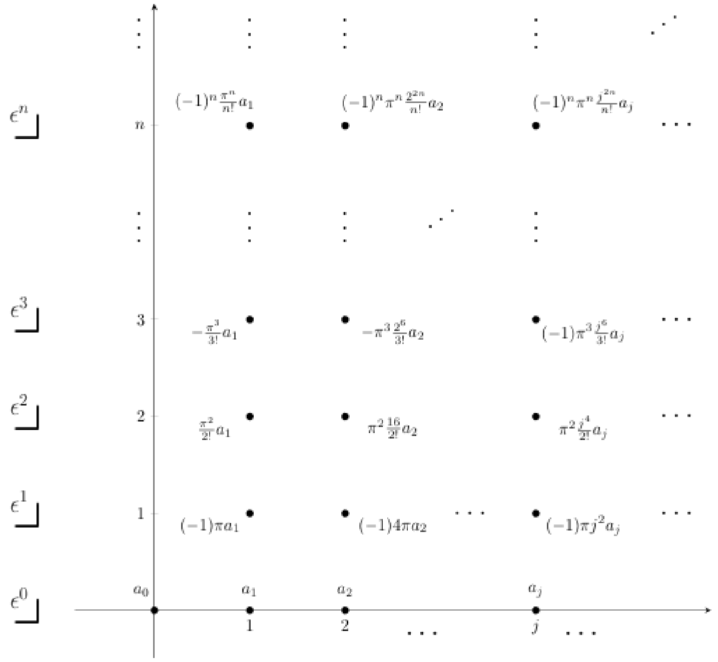
$$f(\epsilon) = \sum_{j=0}^{\infty} a_j e^{-\pi j^2 \epsilon} \tag{35}$$

where $\{a_j\}$ is a periodic coefficient-sequence of period N, so that

$$a_{j+N} = a_j \quad \text{for all } j \tag{36}$$

and where, for simplicity, we shall assume that a_j is indexed in j from $-\infty$ to ∞, even though the sum defining $f(\epsilon)$ only extends from $j = 0$ to $j = \infty$. Expanding

$e^{-\pi j^2 \epsilon}$ in the y-direction using the Taylor series $e^{-\pi j^2 \epsilon} = \sum_{n=0}^{\infty}(-1)^n \pi^n \frac{j^{2n}}{n!}\epsilon^n$ for small ϵ, we thus have the 2-d Césaro array shown in Figure 5 for $f(\epsilon)$:



At height 0 (i.e. order ϵ^0) we get the base sum $\sum_{j=0}^{\infty} a_j$ and the key to making progress is to realise that a_j being periodic is not, in and of itself, quite sufficient to facilitate successful 2-d Césaro array analysis. Rather, we need to adjust these coefficients slightly to make them more strongly periodic, and to allow us thereby to make use of a discrete (i.e. sequence-based not function-based) version of the theory of Césaro-adapted scales developed in [III].

3.1 Césaro-adapted periodic sequences and preliminary results

Background: In [III] we defined the notion of a "Césaro-adapted scale" as a 1-parameter set of functions, each of which is (up to a constant multiple) the anti-derivative of its predecessor, and each of which is Césaro convergent to 0 under a single application of the continuous Césaro operator, P. If the functions are all periodic, this latter condition means that the integral of each function over its period must be 0.

The archetypal example - which we used extensively in [III] and which was crucial in proving various theorems therein - was the set of period-1 functions $\overset{\vee}{q}_n(X) = \overset{\vee}{q}_n(\alpha)$ (where $X = k + \alpha$ as usual) given by $\overset{\vee}{q}_0(\alpha) = \alpha - \frac{1}{2}$, $\overset{\vee}{q}_1(\alpha) = \frac{1}{2}\alpha^2 - \frac{1}{2}\alpha + \frac{1}{12}$, $\overset{\vee}{q}_2(\alpha) = \frac{1}{3}\alpha^3 - \frac{1}{2}\alpha^2 + \frac{1}{6}\alpha$ etc, all of which satisfy $\int_0^1 \overset{\vee}{q}_n(\alpha)\, d\alpha = 0$.

This set was effectively built up by starting with the periodic base function, $\overset{\vee}{q}_0(\alpha)$ satisfying $\int_0^1 \overset{\vee}{q}_0(\alpha)\,d\alpha = 0$, and then creating each new function in the set, $\overset{\vee}{q}_n(\alpha)$, as a multiple of the anti-derivative of the last one ($\overset{\vee}{q}{}'_n(\alpha) = n\overset{\vee}{q}_{n-1}(\alpha)$) subject to the constraint that $\int_0^1 \overset{\vee}{q}_n(\alpha)\,d\alpha = 0$, which fixes the constant of integration.

The power of such a Césaro-adapted scale within Césaro analysis lies in the fact that all products of such functions with powers of X, i.e. all expressions of the form $X^n\overset{\vee}{q}_r(X)$, remain *strongly* Césaro convergent to 0 no matter how large we make n - that is, such products remain Césaro convergent to 0 as $X \to \infty$ under application of a pure power of P, namely P^{n+1}:

$$P^{n+1}[\tilde{X}^n\overset{\vee}{q}_r(\tilde{X})](X) \to 0 \quad \text{classically as } X \to \infty\ . \tag{37}$$

This is because in applying P^{n+1}, repeated integration by parts reduces the power of X by one each time at the expense of increasing successively from $\overset{\vee}{q}_r$ to $\overset{\vee}{q}_{r+1}$, then $\overset{\vee}{q}_{r+2}$ etc in the product - until we are left just with $\overset{\vee}{q}_{r+n}$ and one further application of P makes that function classically convergent to 0.

The discrete case: Now suppose $\{\tilde{a}_j\}_{j=-\infty}^{\infty}$ is a periodic sequence with period N. How does the concept of a Césaro-adapted scale play out in this discrete (sequences) as opposed to continuous (functions) setting? Well, the analogue of derivative is now the discrete (backwards) difference operator, Δ, given in its action on a sequence, $\{s_j\}$, by

$$(\Delta s)_j := s_j - s_{j-1}\ . \tag{38}$$

And, as regards the second condition of Césaro-adapted scales, clearly a period-N sequence, s, converges to zero classically under a single application of the discrete Césaro operator, P_D, if and only if its period sum is zero:

$$P_D[s]_j \to 0 \quad \text{as } j \to \infty \text{ if and only if } \quad \sum_{j=0}^{N-1} s_j = 0 \tag{39}$$

where here P_D is given in the usual way by $P_D[s]_k = \frac{1}{k}\sum_{j=1}^{k} s_j$. We thus make the following definitions:

Definition 1 [Strongly periodic period-N sequences]: *Let the space of sequences, $\mathcal{S}_{[N]}$, be the set of period-N sequences $\{s_j\}_{j=-\infty}^{\infty}$ which also have period-sum* 0*, i.e.*

$$\sum_{j=0}^{N-1} s_j = 0\ . \tag{40}$$

Definition 2 [Discrete Césaro-adapted scales]: *A Césaro-adapted scale within $\mathcal{S}_{[N]}$ is a set of sequences $s^{(n)} \in \mathcal{S}_{[N]}$, $n \in \mathbb{Z}_{\geq 0}$, such that*

$$s_j^{(n)} = s_j^{(n+1)} - s_{j-1}^{(n+1)} \quad \text{for all } j \text{ and } n;\ \text{i.e.} \tag{41}$$

$$s^{(n)} = \Delta s^{(n+1)} \quad \text{as sequences for all } n \in \mathbb{Z}_{\geq 0}\ . \tag{42}$$

Comment: Such a Césaro-adapted scale of sequences is thus a set of sequences satisfying the following three conditions:

(i) That each sequence $s^{(n)}$ has period N ($s^{(n)}_{j+N} = s^{(n)}_j$ for all $j \in \mathbb{Z}$);
(ii) That each sequence $s^{(n)}$ has period-sum 0 ($\sum_{j=0}^{N-1} s^{(n)}_j = 0$); and
(iii) That each sequence $s^{(n)}$, $n \geq 1$, is the discrete anti-derivative of its predecessor, $s^{(n-1)}$.

Just as with the $\overset{\vee}{q}_n(X)$ in [III], if we start with a period-N sequence, $\{\tilde{a}_j\}_{j=-\infty}^{\infty}$, which is in fact in $\mathcal{S}_{[N]}$ (so also satisfies $\sum_{j=0}^{N-1} \tilde{a}_j = 0$), then we can use it as a base-sequence and build a unique Césaro-adapted scale from it having $s^{(0)} = \tilde{a}$. To see this we use the following lemma:

Lemma 1: *Given a sequence* $\{u_j\}_{j=-\infty}^{\infty}$ *in* $\mathcal{S}_{[N]}$, *there exists a unique sequence* $\{u^{(1)}_j\}_{j=-\infty}^{\infty}$ *which is also in* $\mathcal{S}_{[N]}$ *and satisfies*

$$u = \Delta u^{(1)} \ , \quad \text{i.e. } u_j = u^{(1)}_j - u^{(1)}_{j-1} \text{ for all } j \in \mathbb{Z} \ . \tag{43}$$

Proof: Since u is the discrete derivative of $u^{(1)}$, so $u^{(1)}$ should (up to a "constant of integration"), be the anti-derivative of u, which in the discrete setting is of course its partial sum sequence. Let s be this p-sum sequence given by $s_j = \sum_{i=0}^{j} u_i$. Then s is clearly periodic with period N since $s_{N-1} = \sum_{i=0}^{N-1} u_i = 0$ and thus $s_N = u_N = u_0 = s_0$ and $s_{N+1} = u_N + u_{N+1} = u_0 + u_1 = s_1$ and so forth. And it follows by construction that $u = \Delta s$, since for any $j \in \mathbb{Z}$ we have $s_j - s_{j-1} = \sum_{i=0}^{j} u_i - \sum_{i=0}^{j-1} u_i = u_j$. The only thing preventing us from simply taking $u^{(1)}$ as s is thus that s is not necessarily in $\mathcal{S}_{[N]}$, since we cannot guarantee that $\sum_{j=0}^{N-1} s_j = 0$. However, if we let C denote the period-sum of s:

$$C := \sum_{j=0}^{N-1} s_j$$

and we take $u^{(1)}$ as given simply by

$$u^{(1)}_j := s_j - \frac{C}{N} \quad \text{for all } j \in \mathbb{Z}$$

then $u^{(1)}$ remains periodic and trivially still satisfies $u = \Delta u^{(1)}$, but we now clearly have $\sum_{j=0}^{N-1} u^{(1)}_j = 0$. Thus $u^{(1)}$ does lie in $\mathcal{S}_{[N]}$ and represents the unique such sequence in $\mathcal{S}_{[N]}$ meeting the requirements of the lemma. QED

Note: In the above proof, wherever we are taking p-sums such as $\sum_{i=0}^{j} u_i$, if j is a negative index then this is taken, in the spirit outlined in [I]-[III], as a difference of remainder sums, namely $\sum_{i=0}^{j} u_i := R_{+,0}[\{u\}]_0 - R_{+}[\{u\}]_j$. Thus we get $\sum_{i=0}^{-1} u_i = 0$, $\sum_{i=0}^{-2} u_i = -u_{-1} = -u_{N-1}$, $\sum_{i=0}^{-3} u_i = -u_{-1} - u_{-2} = -u_{N-1} - u_{N-2}$ and so on.

Using lemma 1, we can now easily see how to build a Césaro-adapted scale from our given period-N base-sequence $\{\tilde{a}_j\}_{j=-\infty}^{\infty} \in \mathcal{S}_{[N]}$. We simply let $s^{(0)} := \tilde{a}$ and apply lemma 1 to take $s^{(1)}$ as the unique sequence in $\mathcal{S}_{[N]}$ satisfying $\Delta s^{(1)} = s^{(0)}$. We then apply lemma 1 again to take $s^{(2)}$ as the unique sequence in $\mathcal{S}_{[N]}$ satisfying $\Delta s^{(2)} = s^{(1)}$ and so on, iteratively constructing a Césaro-adapted scale of sequences $s^{(n)}$, $n \in \mathbb{Z}_{\geq 0}$, in exact discrete analogy to the way we constructed the continuous Césaro-adapted scale $\overset{\vee}{q}_n(X)$ from the base-function $\overset{\vee}{q}_0(X)$.

Just as, in the continuous setting, all expressions of the form $X^n \overset{\vee}{q}_r(X)$ can be rendered classically convergent to 0 by a sufficiently high power of P; so in the discrete setting we can likewise render any sequence of the form $j^n s_j^{(r)}$ classically convergent to 0 as $j \to \infty$ by applying a sufficiently high power of P_D. Specifically:

Lemma 2: *Suppose $s^{(n)}$, $n \in \mathbb{Z}_{\geq 0}$, is a Césaro-adapted scale within $\mathcal{S}_{[N]}$ and suppose $r, l \in \mathbb{Z}_{\geq 0}$ are arbitrary. Then we have that*

$$P_D^{l+1}\left[\{j^l s_j^{(r)}\}\right]_k \to 0 \quad \text{classically as } k \to \infty \tag{44}$$

i.e. the sequence $\{j^l s_j^{(r)}\}$ is strongly *convergent to* 0 *within the discrete Césaro framework, via the pure power P_D^{l+1}.*

Proof of lemma 2: Proving this key result relies on using summation by parts, and the form of this well-known result which we shall use is as follows:[4]

Theorem [Summation by parts]: *If $\{f_j\}$ and $\{g_j\}$ are any two sequences then we have*

$$\sum_{j=m}^{n} f_j(g_j - g_{j-1}) = [f_n g_n - f_{m-1} g_{m-1}] - \sum_{j=m}^{n} g_{j-1}(f_j - f_{j-1}) \tag{45}$$

or, equivalently,

$$\sum_{j=m}^{n} f_j(\Delta g)_j = [f_n g_n - f_{m-1} g_{m-1}] - \sum_{j=m}^{n} g_{j-1}(\Delta f)_j\ . \tag{46}$$

When $m = 1$ this becomes

$$\sum_{j=1}^{n} f_j(\Delta g)_j = [f_n g_n - f_0 g_0] - \sum_{j=1}^{n} g_{j-1}(\Delta f)_j\ . \tag{47}$$

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[4] The precise form of the statement of summation by parts can be subject to considerable minor variation - for example, equation 47 above is often given alternatively as $\sum_{j=1}^{n} f_j(\Delta g)_j = [f_{n+1} g_n - f_1 g_0] - \sum_{j=1}^{n} g_j(\Delta f)_{j+1}$.

Using this, our approach to proving lemma 2 is then morally clear. When applying P_D, we use summation by parts to reduce the divergent power, j^l, by one each time at the expense of successively moving up the rungs of the Césaro-adapted scale from $s^{(r)}$ to $s^{(r+1)}$, then $s^{(r+2)}$ and so on until we reach $j^0 s^{(r+l)}$. We then use one further power of P_D to render this classically convergent to 0, since $s^{(r+l)}$ has period-sum 0.

In carrying out this program, we use induction to render the argument cleaner and avoid too much iterative mess. However, it will be best to work indirectly, using one further insight to simplify the mechanics of the proof rather than plunging straight in.

A simplifying observation: Consider the case of $X^n \overset{\vee}{q}_r(X)$ in the continuous Césaro setting, and the proof that $P^{n+1}[\tilde{X}^n \overset{\vee}{q}_r(\tilde{X})](X) \to 0$ classically. This works particularly cleanly because not only are the functions in the Césaro-adapted scale, $\overset{\vee}{q}_{r+m}$, all related neatly by differentiation/anti-differentiation, but when we apply integration by parts and flip the application of the derivative to the X^{n-m} term during each application of P, we move equally cleanly down to the next power down, X^{n-m-1}. In other words, both the powers, X^{n-m}, *and* the Césaro-adapted scale form ladders in which differentiation/anti-differentiation move us directly up and down the rungs, without any messy extra terms requiring attention each time we apply P.

In the same way, it will be best if instead of using pure power-sequences $\{j^l\}$ in our products with the members of our Césaro-adapted scale, $s^{(r)}$, we instead use a ladder of similar sequences which are related neatly by the application of the discrete derivative operator, Δ, in the same way that Δ moves us cleanly down the rungs of the Césaro-adapted scale. That way, our use of summation by parts in applying P_D will be clean and will allow us to formulate a simple inductive proof.

What should this alternative ladder of "power-like" sequences be? Well, just as the X^n were non-trivial eigensequences of P in the continuous Césaro context, so here our alternative ladder should consist of non-trivial eigensequences of P_D. In light of our findings in [I], we thus here take the binomial eigensequences $\{\binom{j}{l}\}_{j=-\infty}^{\infty}$ and instead first prove the following variant of lemma 2:

Lemma 3: *Suppose* $s^{(n)}$, $n \in \mathbb{Z}_{\geq 0}$, *is a Césaro-adapted scale within* $\mathcal{S}_{[N]}$ *and suppose* $r, l \in \mathbb{Z}_{\geq 0}$ *are arbitrary. Then we have that the sequence* $\{\binom{j}{l} s_j^{(r)}\}_{j=-\infty}^{\infty}$ *is* strongly *Césaro convergent to* 0 *within the discrete Césaro framework via the pure power* P_D^{l+1}, *i.e.*

$$P_D^{l+1}\left[\left\{\binom{j}{l} s_j^{(r)}\right\}\right]_k \to 0 \quad \text{classically as } k \to \infty \,. \tag{48}$$

Proof of lemma 3 by induction on l: In the base case $l = 0$ the result follows immediately because $s^{(r)}$ is part of a Césaro-adapted scale (it has period-sum 0 and so converges to 0 under P_D).

Now suppose the result holds true for all $l \leq L$. Then for $l = L+1$, using summation by parts and noting that

$$\left(\Delta\left\{\binom{\tilde{j}}{L+1}\right\}\right)_j = \binom{j}{L+1} - \binom{j-1}{L+1} = \binom{j-1}{L} ,$$

we have that

$$\begin{aligned}
P_D\left[\left\{\binom{j}{L+1}s_j^{(r)}\right\}\right]_k &= \frac{1}{k}\sum_{j=1}^{k}\binom{j}{L+1}s_j^{(r)} = \frac{1}{k}\sum_{j=1}^{k}\binom{j}{L+1}\left(\Delta s^{(r+1)}\right)_j \\
&= \frac{1}{k}\left\{\binom{k}{L+1}s_k^{(r+1)} - \sum_{j=1}^{k}s_{j-1}^{(r+1)}\left(\Delta\binom{\tilde{j}}{L+1}\right)_j\right\} \\
&= \frac{1}{k}\left\{\binom{k}{L+1}s_k^{(r+1)} - \sum_{j=1}^{k}s_{j-1}^{(r+1)}\binom{j-1}{L}\right\} \\
&= \frac{1}{k}\left\{\binom{k}{L+1}s_k^{(r+1)} + \binom{k}{L}s_k^{(r+1)} - \binom{0}{L}s_0^{(r+1)} - \sum_{j=1}^{k}s_j^{(r+1)}\binom{j}{L}\right\} \\
&= \frac{1}{k}\binom{k+1}{L+1}s_k^{(r+1)} - P_D\left[\left\{\binom{j}{L}s_j^{(r+1)}\right\}\right]_k + o(1) \, . \qquad (49)
\end{aligned}$$

Now $\binom{k+1}{L+1}$ is a polynomial of degree $L+1$ and so, modulo a $o(1)$ residual term, $\frac{1}{k}\binom{k+1}{L+1}$ is a polynomial of degree L. But the sequences $\binom{j}{0}$, $\binom{j}{1}$, ..., $\binom{j}{L}$ form a basis spanning all polynomials of degree L. Thus we can write $\frac{1}{k}\binom{k+1}{L+1}s_k^{(r+1)}$ as a linear combination

$$\frac{1}{k}\binom{k+1}{L+1}s_k^{(r+1)} = \sum_{i=0}^{L}c_i\binom{k}{i}s_k^{(r+1)} + o(1)$$

for some collection of constants $c_0, c_1, \ldots, c_L$. It then follows in equation 49, by our inductive hypothesis and the regularity of P_D, that

$$\begin{aligned}
P_D^{L+2}\left[\left\{\binom{j}{L+1}s_j^{(r)}\right\}\right]_k &= \left\{\begin{array}{c} \sum_{i=0}^{L}c_i P_D^{L+1}\left[\left\{\binom{j}{i}s_j^{(r+1)}\right\}\right]_k \\ -P_D^{L+2}\left[\left\{\binom{j}{L}s_j^{(r+1)}\right\}\right]_k + o(1) \end{array}\right\} \\
&\to 0 \quad \text{classically as } k \to \infty \, ,
\end{aligned}$$

so that lemma 3 also holds for $l = L+1$. This completes the inductive step and hence the proof of lemma 3 for all $l, r \in \mathbb{Z}_{\geq 0}$. QED.

Completing the proof of lemma 2: With lemma 3 in hand it is now trivial to go back and prove lemma 2. We simply observe, as above, that the degree l polynomial, j^l, appearing in lemma 2 can be written as a linear combination of binomial expressions $\binom{j}{i}, 0 \leq i \leq l$. Invoking lemma 3 for each of the terms arising from the linear combination, the result follows at once. QED.

So far, in this section, we have been developing a collection of preliminary results which we will need in order to conduct 2-d Césaro array analysis for any array of the sort shown in Figure 5 - of which the arrays for $\tilde{G}(z)$ when $z^2 = -\frac{2i}{p} + \epsilon$ or $z^2 = -\frac{i}{p} + \epsilon$ or $z^2 = -\frac{ik}{p} + \epsilon$ in general, are examples. There is just one more such preliminary we need. This is a result giving the generalised discrete Césaro value of a sum $\sum_{j=0}^{\infty} \tilde{a}_j$ in the case where $\{\tilde{a}_j\}_{j=-\infty}^{\infty}$ is in $\mathcal{S}_{[N]}$ and also satisfies the further symmetry condition that $\tilde{a}_{N-j} = \tilde{a}_j$ for all $j = 0, 1, \ldots, N-1$. The result is as follows:

Result 1: *Suppose* $\{\tilde{a}_j\}_{j=-\infty}^{\infty}$ *is a sequence in* $\mathcal{S}_{[N]}$ *satisfying the symmetry condition that* $\tilde{a}_{N-j} = \tilde{a}_j$ *for all* $j = 0, 1, \ldots, N-1$ *(and hence for all* j*). Then*

$$\sum_{j=0}^{\infty} \tilde{a}_j = \frac{\tilde{a}_0}{2} \tag{50}$$

in a generalised discrete Césaro sense, via the regular polynomial $q(P_D) = P_D$*; i.e. the p-sum sequence* $s_j := \sum_{i=0}^{j} \tilde{a}_i$ *for this series satisfies that*

$$P_D\left[\{s_j\}\right]_k \to \frac{\tilde{a}_0}{2} \tag{51}$$

classically as $k \to \infty$*.*

Proof: Since $\{\tilde{a}_j\}$ is in $\mathcal{S}_{[N]}$ it satisfies $\sum_{j=0}^{N-1} \tilde{a}_j = 0$, i.e. $s_{N-1} = 0$, and so the p-sum sequence $\{s_j\}$ is also periodic with period N. Now the discrete Césaro limit of a periodic sequence is obviously given, under a single application of P_D, by its average value over a cycle. Thus

$$\begin{aligned} C_D \underset{k\to\infty}{lim}\, s_k &= \frac{1}{N}\sum_{j=0}^{N-1} s_j = \frac{1}{N}\sum_{j=0}^{N-1}\left\{\sum_{i=0}^{j}\tilde{a}_i\right\} \\ &= \frac{1}{N}\left\{N\cdot\tilde{a}_0 + (N-1)\cdot\tilde{a}_1 + \ldots + 2\cdot\tilde{a}_{N-2} + 1\cdot\tilde{a}_{N-1}\right\}\ . \end{aligned}$$

But since $\tilde{a}_j$ satisfies that $\tilde{a}_{N-j} = \tilde{a}_j$ for all j, this can be re-expressed as

$$\begin{aligned} C_D \underset{k\to\infty}{lim}\, s_k &= \frac{1}{2N}\left\{\begin{array}{c} N\cdot\tilde{a}_0 + (N-1)\cdot\tilde{a}_1 + \ldots + 1\cdot\tilde{a}_{N-1} + 0\cdot\tilde{a}_N + \\ 0\cdot\tilde{a}_0 + 1\cdot\tilde{a}_1 + \ldots + (N-1)\cdot\tilde{a}_{N-1} + N\cdot\tilde{a}_N \end{array}\right\} \\ &= \frac{1}{2}\left\{\sum_{j=0}^{N-1}\tilde{a}_j + \tilde{a}_N\right\} = \frac{1}{2}\tilde{a}_N = \frac{1}{2}\tilde{a}_0 \end{aligned}$$

and this completes the proof of the result. QED.

With result 1 and our previous lemmas we are now in a position to conduct our 2-d Césaro array analysis of the array in Figure 5 and hence read off the promised results for finite exponential sums; as well as extend our singularity analysis on the boundary of $\tilde{G}(z)$ to the more general case of $z^2 = -\frac{ik}{p} + \epsilon$, where our previous methods in section 2 began to encounter difficulties.

3.2 2-d Césaro array results, evaluation of finite exponential sums and extended boundary singularity analysis for $\tilde{G}(z)$

Figure 5 provides the 2-d Césaro array for $f(\epsilon) := \sum_{j=0}^{\infty} a_j e^{-\pi j^2 \epsilon}$, where $\{a_j\}$ is a period-N sequence. Using the methodology developed in [IV] for the analysis of such 2-d Césaro arrays, we consider a succession of cases.

Case 1 [$\{a_j\}$ is in fact in $\mathcal{S}_{[N]}$]: If $\{a_j\}$ is in $\mathcal{S}_{[N]}$ and so also satisfies $\sum_{j=0}^{N-1} a_j = 0$, then it follows immediately that the horizontal p-sum sequence at height 0 (i.e. at order ϵ^0) is periodic and so converges in a discrete Césaro sense to its limiting sum A_0 via a single application of P_D. Moreover, $\{a_j\}$ can be taken as the base-sequence in a Césaro-adapted scale, $s^{(n)}$ (i.e. $a_j = s_j^{(0)}$).

Now, at any height l (i.e. order ϵ^l), we can express j^{2l} as a linear combination of binomial eigensequences $\binom{j}{2l}, \binom{j}{2l-1}, \ldots, \binom{j}{1}$ and $\binom{j}{0}$. It follows, by summation by parts in the same manner as applied in proving lemmas 2 and 3, that the horizontal p-sum sequence at height l can be expressed also as some constant A_l (which will be its discrete Césaro limit) plus a linear combination of products of the form $\binom{j}{m} s_j^{(r)}$ where $0 \leq m, r \leq 2l$ and $0 \leq r + m \leq 2l$. But then it follows immediately by lemma 3 that all the non-constant terms in this linear combination have generalised discrete Césaro limit 0, via a pure power of P_D (in fact, we have $0 \leq r + m \leq 2l$, so we can take the power as P_D^{2l+1}).

Thus in our Césaro array, all the horizontal sums converge in the discrete Césaro framework via pure powers of P_D. The critical corollary of this is that we have no "second-component" pieces arising from these horizontal sums to re-combine vertically, and thus no additional piece to include in our 2-d Césaro array analysis for $f(\epsilon)$. We have thus proven that:

Result 2: *If $\{a_j\}_{j=-\infty}^{\infty}$ is in $\mathcal{S}_{[N]}$, then $f(\epsilon) := \sum_{j=0}^{\infty} a_j e^{-\pi j^2 \epsilon}$ is given by the pure power series*

$$f(\epsilon) = \sum_{n=0}^{\infty} A_n \epsilon^n + \mathcal{S}_0(\epsilon) \quad \text{as } \epsilon \to 0^+ \tag{52}$$

where each A_n is the degree-wise discrete Césaro sum of $\sum_{j=0}^{\infty} (-1)^n \frac{\pi^n}{n!} j^{2n} a_j$ via P_D^{2n+1}; there is no extra singular term arising from re-combination of second-component pieces.

Comment: It is easy (I've done it repeatedly!) to slip into the error of believing that in this derivation the A_n, $n \geq 0$, are all identically zero, based on lemmas 2 and 3. however, This is not true! Just to take one example, in Result 1 we have seen that if $\{a_j\}$ is not just in $\mathcal{S}_{[N]}$ but also satisfies the symmetry condition $a_{N-j} = a_j$, then $A_0 = \frac{1}{2}a_0 \neq 0$ generically.

The key point is that while lemmas 2 and 3 show that *sequences* of the form $j^l s_j^{(r)}$ or $\binom{j}{l} s_j^{(r)}$ converge to 0 in a generalised discrete Césaro sense under P_D^{l+1}, we first need to apply *summation by parts* at level n to obtain our *p-sum* sequence from the terms $(-1)^n \frac{\pi^n}{n!} j^{2n} a_j$ before taking Césaro limits. And while this process creates a final sequence consisting mainly of pieces of the desired form (i.e. $\binom{j}{m} s_j^{(r)}$) with discrete Césaro limit 0, it also generates a collection of constant terms involving $s_0^{(r)}$ which combine together to give A_n - in the same way that in integration by parts on $[0, X]$ we pick up limiting constant evaluations at the left-hand end-point of the integration interval, 0, along with the variable terms from evaluation at the right-hand end-point X. The power series in equation 52 is thus not, in general, trivial (mod $\mathcal{S}_0(\epsilon)$)

Interestingly, however, we will shortly see that in the cases which we have been pre-occupied with so far in this paper, namely those considered in section 2, we *do* in fact have all the A_n, $n \in \mathbb{Z}_{>0}$ being identically zero! We shall briefly comment on this when we get to these cases later in this section. We shall also return briefly to this general case in section 4, where we outline how the calculation of the A_n in general can be systematised, and give conditions under which they are identically zero.

Case 2 [$\{a_j\}$ is periodic but not in $\mathcal{S}_{[N]}$]: If $\{a_j\}$ has period N but is not in $\mathcal{S}_{[N]}$, then $\sum_{j=0}^{N-1} a_j \neq 0$. Let $\sigma := \sum_{j=0}^{N-1} a_j$. Then we have seen that the adjusted sequence $\{\tilde{a}_j\}_{j=-\infty}^{\infty}$ given by

$$\tilde{a}_j := a_j - \frac{\sigma}{N} \tag{53}$$

is in $\mathcal{S}_{[N]}$. It follows from case 1 that, using the obvious notation, we have

$$\sum_{j=0}^{\infty} \tilde{a}_j e^{-\pi j^2 \epsilon} = \sum_{n=0}^{\infty} \tilde{A}_n \epsilon^n + \mathcal{S}_0(\epsilon) \quad , \tag{54}$$

while we also know from [IV] that

$$\begin{aligned} \sum_{j=0}^{\infty} \frac{\sigma}{N} e^{-\pi j^2 \epsilon} &= \frac{\sigma}{2N} + \frac{\sigma}{N} \left\{ \frac{1}{2} + \sum_{j=1}^{\infty} e^{-\pi j^2 \epsilon} \right\} \\ &= \frac{\sigma}{2N} + \frac{\sigma}{2N} \frac{1}{\sqrt{\epsilon}} + \mathcal{S}_0(\epsilon) \; . \end{aligned} \tag{55}$$

Since $a_j = \tilde{a}_j + \frac{\sigma}{N}$, we have shown the following:

Result 3: *If* $\{a_j\}_{j=-\infty}^{\infty}$ *is periodic with period* N*, and* $\sigma := \sum_{j=0}^{N-1} a_j$ *then* $f(\epsilon) := \sum_{j=0}^{\infty} a_j e^{-\pi j^2 \epsilon}$ *is given by*

$$f(\epsilon) = \frac{\sigma}{2N}\frac{1}{\sqrt{\epsilon}} + \left(\frac{\sigma}{2N} + \tilde{A}_0\right) + \sum_{n=1}^{\infty} \tilde{A}_n \epsilon^n + \mathcal{S}_0(\epsilon) \quad \text{as } \epsilon \to 0^+ \tag{56}$$

where each $\tilde{A}_n$ *is the degree-wise Césaro sum via* P_D^{2n+1} *of* $\sum_{j=0}^{\infty} (-1)^n \frac{\pi^n}{n!} j^{2n} \tilde{a}_j$ *and* $\{\tilde{a}_j\} \in \mathcal{S}_{[N]}$ *is the adjusted sequence given by* $\tilde{a}_j = a_j - \frac{\sigma}{N}$*.*

Case 3 [$\{a_j\}$ is periodic with period N and satisfies $a_{N-j} = a_j$ for all j]: This is the same as case 2 except that we also have the symmetry condition $a_{N-j} = a_j$. Since $\tilde{a}_j = a_j - \frac{\sigma}{N}$ also clearly satisfies this condition, it follows from Result 1 applied at height 0 that we can evaluate $\tilde{A}_0$ explicitly as

$$\tilde{A}_0 = \frac{1}{2}\tilde{a}_0 = \frac{1}{2}a_0 - \frac{\sigma}{2N}$$

and thus in this case Result 3 simplifies to

Result 4: *If* $\{a_j\}_{j=-\infty}^{\infty}$ *is periodic with period* N, $\sigma := \sum_{j=0}^{N-1} a_j$*, and* $a_{N-j} = a_j$ *for all* j *then* $f(\epsilon) := \sum_{j=0}^{\infty} a_j e^{-\pi j^2 \epsilon}$ *is given by*

$$f(\epsilon) = \frac{\sigma}{2N}\frac{1}{\sqrt{\epsilon}} + \frac{a_0}{2} + \sum_{n=1}^{\infty} \tilde{A}_n \epsilon^n + \mathcal{S}_0(\epsilon) \quad \text{as } \epsilon \to 0^+ \tag{57}$$

where each $\tilde{A}_n$, $n \in \mathbb{Z}_{\geq 1}$*, is given as in Result 3.*

3.2.1 Application to $\tilde{G}(z)$ and finite exponential sums

With these results in hand we can now return to the particular cases that animated our interest in studying $\tilde{G}(z)$ in section 2, and thereby quickly read off the promised initial results regarding finite exponential sums.

Case 4 [$f(\epsilon) := \frac{1}{2} + \sum_{j=1}^{\infty} e^{2\pi i \frac{j^2}{p}} e^{-\pi j^2 \epsilon}$ where $p \in \mathbb{Z}_{>0}$ is odd]: This case corresponds to our study in section 2 of $\tilde{G}(z)$ as $z \to \nu_{2,p}$ so that $z^2 = -\frac{2i}{p} + \epsilon$ (see equation 5). We showed in that section (see equation 22) that we have

$$f(\epsilon) = \begin{cases} \frac{1}{2\sqrt{p}}\frac{1}{\sqrt{\epsilon}} + \mathcal{S}_0(\epsilon) & , p \equiv 1 \ (mod\ 4) \\ \\ \frac{i}{2\sqrt{p}}\frac{1}{\sqrt{\epsilon}} + \mathcal{S}_0(\epsilon) & , p \equiv 3 \ (mod\ 4) \end{cases} \tag{58}$$

But letting $a_j = e^{2\pi i \frac{j^2}{p}}$ we see that $\{a_j\}_{j=-\infty}^{\infty}$ is periodic with period p (i.e. $N = p$) and satisfies the symmetry condition $a_{p-j} = a_j$; and that the series defining $f(\epsilon)$ is precisely $\sum_{j=0}^{\infty} a_j e^{-\pi j^2 \epsilon}$ except that the term at $j = 0$ is not

$a_0 = 1$ but instead $\frac{1}{2}$. It follows from equation 57 in case 3 that we have, alternatively,

$$f(\epsilon) = \frac{\sigma}{2p}\frac{1}{\sqrt{\epsilon}} + \sum_{n=1}^{\infty} \tilde{A}_n \epsilon^n + \mathcal{S}_0(\epsilon) \tag{59}$$

where $\sigma := \sum_{j=0}^{p-1} e^{2\pi i \frac{j^2}{p}}$ and each $\tilde{A}_n$, $n \in \mathbb{Z}_{\geq 1}$, is the degree-wise Césaro sum via P_D^{2n+1} of $\sum_{j=1}^{\infty}(-1)^n \frac{\pi^n}{n!} j^{2n}\left(e^{2\pi i \frac{j^2}{p}} - \frac{\sigma}{p}\right)$.

Comparing equations 58 and 59 we can then immediately read off a number of corollaries including, as a first evaluation of a finite exponential sum, the following result due to Gauss:

Corollary 1: *We have*

$$\sum_{j=0}^{p-1} e^{2\pi i \frac{j^2}{p}} = \begin{cases} \sqrt{p} & , \ p \equiv 1 \ (mod\ 4) \\ \\ i\sqrt{p} & , \ p \equiv 3 \ (mod\ 4)\,. \end{cases} \tag{60}$$

Notes: **(i)** This result is usually quoted just for p prime, but our only requirement has been that p be odd and it holds in general for such p.

(ii) Corollary 1 only requires comparison of the singular terms in equations 58 and 59. Comparing the terms at order ϵ^n, $n \in \mathbb{Z}_{\geq 1}$ these are all zero in equation 58 and it follows in equation 59 that we must also have $\tilde{A}_n = 0$ for all $n \in \mathbb{Z}_{\geq 1}$, i.e.

Corollary 2: *For all $n \in \mathbb{Z}_{\geq 1}$ we have the discrete Césaro sum*

$$\sum_{j=1}^{\infty} j^{2n}\left(e^{2\pi i \frac{j^2}{p}} - \frac{\sigma}{p}\right) = 0 \tag{61}$$

via P_D^{2n+1} (where $\sigma = \sum_{j=0}^{p-1} e^{2\pi i \frac{j^2}{p}}$ is as given in corollary 1).

Comments: **(i)** Corollary 2 shows that our 2-d Césaro array analysis in this section, when combined with the 1-d Césaro analysis of $\tilde{G}(z)$ near $\nu_{2,p}$ in section 2, not only leads to the evaluation of the finite exponential sum in corollary 1; it also furnishes a sort of generating function, yielding the extra countable collection of identities given in corollary 2. It is true that these are identities regarding discrete Césaro limits of p-sum sequences (i.e. for $C_D lim_{k\to\infty} \sum_{j=1}^{k} j^{2n}\left(e^{2\pi i \frac{j^2}{p}} - \frac{\sigma}{p}\right)$), rather than explicit evaluations of finite sums $\sum_{j=0}^{p-1}$, but this is only natural given that the extra j^{2n}-dependence in the terms is not periodic. In section 4 we will return to such identities in general.

(ii) Recall that in [I] we actually calculated $\sum_{j=1}^{\infty} j^m$ under the discrete Césaro scheme as

$$\sum_{j=1}^{\infty} j^m \;=\; C_D\lim_{k\to\infty} \sum_{j=1}^{k} j^m \;=\; 1 \quad \text{for all } m \in \mathbb{Z}_{\geq 0} \tag{62}$$

albeit that, for any given m, the Césaro limit is obtained via a somewhat complicated regular polynomial $q(P_D)$ - one which involves factors $\left(\frac{r+1}{r}\right)\left(P_D - \frac{1}{r+1}\right)$ for each $1 \leq r \leq m+1$, as required in order to annihilate eigensequences $\binom{k-1}{r}$ in the p-sum sequence. It follows that if we split the sum in corollary 2, then we can re-express it as an equivalent corollary for sums not involving the strongly-periodic sequence $\tilde{a}_j = e^{2\pi i \frac{j^2}{p}} - \frac{\sigma}{p}$ in $\mathcal{S}_{[p]}$, but rather just the "raw" sequence $a_j = e^{2\pi i \frac{j^2}{p}}$, namely:

Corollary 2a: *For all $n \in \mathbb{Z}_{\geq 1}$ we have the discrete Césaro sum*

$$\sum_{j=1}^{\infty} j^{2n} e^{2\pi i \frac{j^2}{p}} = \frac{\sigma}{p} = \begin{cases} \frac{1}{\sqrt{p}} & , \; p \equiv 1 \; (mod\; 4) \\ \\ \frac{i}{\sqrt{p}} & , \; p \equiv 3 \; (mod\; 4) \end{cases} \tag{63}$$

via the regular polynomial $q(P_D) = P_D^{2n+1} \cdot \prod_{r=1}^{2n+1} \left(\frac{r+1}{r}\right)\left(P_D - \frac{1}{r+1}\right)$.

Case 5 [$f(\epsilon) := \frac{1}{2} + \sum_{j=1}^{\infty} e^{i\pi \frac{j^2}{p}} e^{-\pi j^2 \epsilon}$ where $p \in \mathbb{Z}_{>0}$ is even]: This corresponds to our study in section 2 of $\tilde{G}(z)$ as $z \to \nu_{1,p}$ so that $z^2 = -\frac{i}{p} + \epsilon$ where p is even. There we showed in equation 30 that

$$f(\epsilon) = \frac{e^{i\frac{\pi}{4}}}{2\sqrt{p}} \frac{1}{\sqrt{\epsilon}} + \mathcal{S}_0(\epsilon) \quad \text{as} \quad \epsilon \to 0^+ \; . \tag{64}$$

Letting $a_j = e^{i\pi \frac{j^2}{p}}$, it follows from the fact that p is even that $\{a_j\}_{j=-\infty}^{\infty}$ is again periodic with period p $(N = p)$ and satisfies $a_{p-j} = a_j$, and so it follows exactly as in case 4 that we alternatively have

$$f(\epsilon) = \frac{\sigma}{2p} \frac{1}{\sqrt{\epsilon}} + \sum_{n=1}^{\infty} \tilde{A}_n \epsilon^n + \mathcal{S}_0(\epsilon) \quad \text{as} \quad \epsilon \to 0^+ \tag{65}$$

where $\sigma := \sum_{j=0}^{p-1} e^{i\pi \frac{j^2}{p}}$ and each $\tilde{A}_n$, $n \in \mathbb{Z}_{\geq 1}$, is the degree-wise Césaro sum via P_D^{2n+1} of $\sum_{j=1}^{\infty} (-1)^n \frac{\pi^n}{n!} j^{2n} \left(e^{i\pi \frac{j^2}{p}} - \frac{\sigma}{p}\right)$. Comparing equations 64 and 65 it follows in this case that

Corollary 3: *When $p \in \mathbb{Z}_{>0}$ is even we have*

$$\sigma := \sum_{j=0}^{p-1} e^{i\pi \frac{j^2}{p}} = e^{i\frac{\pi}{4}} \cdot \sqrt{p} \tag{66}$$

and for all $n \in \mathbb{Z}_{\geq 1}$ *we have the discrete Césaro sums*

$$\sum_{j=1}^{\infty} j^{2n} \left(e^{i\pi \frac{j^2}{p}} - \frac{e^{i\frac{\pi}{4}}}{\sqrt{p}} \right) = 0 \tag{67}$$

via P_D^{2n+1}, *which implies that we also have the "raw" discrete Césaro sums*

$$\sum_{j=1}^{\infty} j^{2n} e^{i\pi \frac{j^2}{p}} = \frac{e^{i\frac{\pi}{4}}}{\sqrt{p}} \tag{68}$$

via the regular polynomials $q(P_D) = P_D^{2n+1} \cdot \prod_{r=1}^{2n+1} \left(\frac{r+1}{r}\right) \left(P_D - \frac{1}{r+1}\right)$.

Case 6 [$f(\epsilon) := \frac{1}{2} + \sum_{j=1}^{\infty} e^{i\pi \frac{j^2}{p}} e^{-\pi j^2 \epsilon}$ **where** $p \in \mathbb{Z}_{>0}$ **is odd]:** This is the same as case 5 but with p odd, and in section 2 we showed that then

$$f(\epsilon) = \mathcal{S}_0(\epsilon) \quad \text{as} \quad \epsilon \to 0^+ \; . \tag{69}$$

Since p is odd the coefficients $a_j = e^{i\pi \frac{j^2}{p}}$ are now periodic with period $2p$ ($N = 2p$) and satisfy $a_{2p-j} = a_j$, and so here we have that alternatively

$$f(\epsilon) = \frac{\sigma}{4p} \frac{1}{\sqrt{\epsilon}} + \sum_{n=1}^{\infty} \tilde{A}_n \epsilon^n + \mathcal{S}_0(\epsilon) \quad \text{as} \quad \epsilon \to 0^+ \tag{70}$$

where $\sigma := \sum_{j=0}^{2p-1} e^{i\pi \frac{j^2}{p}}$ and each $\tilde{A}_n$, $n \in \mathbb{Z}_{\geq 1}$, is the degree-wise Césaro sum via P_D^{2n+1} of $\sum_{j=1}^{\infty} (-1)^n \frac{\pi^n}{n!} j^{2n} \left(e^{i\pi \frac{j^2}{p}} - \frac{\sigma}{2p} \right)$. Comparing equations 69 and 70, in this case it follows that:

Corollary 4: *When* $p \in \mathbb{Z}_{>0}$ *is odd we have*

$$\sigma := \sum_{j=0}^{2p-1} e^{i\pi \frac{j^2}{p}} = 0 \tag{71}$$

and for all $n \in \mathbb{Z}_{\geq 1}$ *we have the discrete Césaro sums*

$$\sum_{j=1}^{\infty} j^{2n} e^{i\pi \frac{j^2}{p}} = 0 \tag{72}$$

via P_D^{2n+1}.

Note: The fact that $\sum_{j=0}^{2p-1} e^{i\pi \frac{j^2}{p}} = 0$ when p is odd, is in fact immediately clear without having to do any Césaro analysis, since for all $j = 0, 1, \ldots, (p-1)$ we have $e^{i\pi \frac{(p+j)^2}{p}} = e^{i\pi p} \cdot e^{2\pi i j} \cdot e^{i\pi \frac{j^2}{p}} = -e^{i\pi \frac{j^2}{p}}$ so that we get pairwise cancellation of terms in the sum. There was no such simplification available in our previous cases 4 and 5. In terms of our 2-d Césaro analysis for $f(\epsilon)$, the fact that $\sum_{j=0}^{2p-1} e^{i\pi\frac{j^2}{p}} = 0$ means that $\{a_j\}_{j=-\infty}^{\infty}$ is in fact in $\mathcal{S}_{[2p]}$ directly without any need for adjustment. This is why everything in our analysis collapses to have no singular term and to require only pure powers of P_D - because we do not have to subtract off and re-add an array corresponding to a multiple $(\frac{\sigma}{N})$ of $\frac{1}{2} + \sum_{j=1}^{\infty} e^{-\pi j^2\epsilon}$, with its attendant singular piece and non-trivial eigensequences in each horizontal sum.

Case 7 [$f(\epsilon) := \frac{1}{2} + \sum_{j=1}^{\infty} e^{i\pi k\frac{j^2}{p}} e^{-\pi j^2\epsilon}$ where $k, p \in \mathbb{Z}_{>0}$ and k and p are co-prime]: This corresponds to our general study in section 2 of the behaviour of $\tilde{G}(z)$ as $z \to \nu_{k,p}$ so that $z^2 = -\frac{ik}{p} + \epsilon$; but where we only require k and p to be co-prime, so that such points, $\nu_{k,p}$, lie densely along the entire lower boundary of the region of convergence of $\tilde{G}(z)$, not just near to 0. In this case, however, we were unable to make any progress for $k \geq 3$ in section 2 using 1-d Césaro analysis coupled with the functional equation for $\tilde{G}$ and Césaro dilation-invariance. Can we instead make progress using the methods of this section?

The answer is yes. The key is that $a_j = e^{i\pi k\frac{j^2}{p}}$ is still periodic with period either $N = p$ (if one of k or p is even) or $N = 2p$ (if k and p are both odd) and still satisfies $a_{N-j} = a_j$. Thus result 4 applies and gives us (after adjusting for the $j = 0$ term being $\frac{1}{2}$ rather than 1):

$$f(\epsilon) = \frac{\sigma}{2N}\frac{1}{\sqrt{\epsilon}} + \sum_{n=1}^{\infty} \tilde{A}_n\epsilon^n + \mathcal{S}_0(\epsilon) \quad \text{as} \quad \epsilon \to 0^+ \tag{73}$$

where $\sigma := \sum_{j=0}^{N-1} e^{i\pi k\frac{j^2}{p}}$ and each $\tilde{A}_n$, $n \in \mathbb{Z}_{\geq 1}$, is the degree-wise Césaro sum via P_D^{2n+1} of $\sum_{j=1}^{\infty}(-1)^n \frac{\pi^n}{n!} j^{2n}\left(e^{i\pi k\frac{j^2}{p}} - \frac{\sigma}{N}\right)$.

In the cases of $k = 1$ and $k = 2$ dealt with in cases 4-6 above, we had calculated the power series expansion for $f(\epsilon)$ independently in section 2 and used comparison with these expansions to read off the value of the finite exponential sum σ, and also to deduce Césaro results regarding $\tilde{A}_n$, $n \in \mathbb{Z}_{\geq 1}$. For $k \geq 3$ we reverse perspective. We try to evaluate σ independently - either exactly or at least whether it is zero or non-zero - and hence deduce, in equation 73, whether $f(\epsilon)$ has a singular term or not and thus whether $\nu_{k,p}$ is a zero of $\tilde{G}$ or a square-root branch-point singularity.

Sub-case (i) [k and p both odd] If k, p are both odd then $N = 2p$ and (as for $k = 1$ and $p = 1$ in case 6 above), we have that for all $j = 0, 1, \ldots, (p-1)$, $e^{i\pi k\frac{(p+j)^2}{p}} = e^{i\pi kp} \cdot e^{2\pi ikj} \cdot e^{i\pi k\frac{j^2}{p}} = -e^{i\pi k\frac{j^2}{p}}$ so that we get pairwise cancellation and can immediately conclude that $\sigma = 0$. It follows at once in equation 73 that:

Corollary 5: *When $k, p \in \mathbb{Z}_{>0}$ are both odd we have*

$$f(\epsilon) = \sum_{n=1}^{\infty} A_n \epsilon^n + \mathcal{S}_0(\epsilon) \quad \text{as} \quad \epsilon \to 0^+ \tag{74}$$

where each A_n, $n \in \mathbb{Z}_{\geq 1}$, is the degree-wise discrete Césaro sum via P_D^{2n+1} of $\sum_{j=1}^{\infty} (-1)^n \frac{\pi^n}{n!} j^{2n} e^{i\pi k \frac{j^2}{p}}$. In particular, $\nu_{k,p} = \sqrt{\frac{-ik}{p}}$ is a zero of $\tilde{G}(z)$ on its lower boundary.

Comments: **(a)** Note that for $k \geq 3$ we cannot immediately conclude that $\nu_{k,p}$ is a Schwartzian zero of $\tilde{G}(z)$, as we did when $k = 1$ in case 6. This is because we do not have a result from section 2 to compare with and we have not separately evaluated the discrete Césaro sums $\sum_{j=1}^{\infty} j^{2n} e^{i\pi k \frac{j^2}{p}}$. We say more on this in section 4.

(b) Points of the form $\nu_{k,p}$ with k, p both odd and co-prime clearly fill up the lower boundary (and correspondingly the upper boundary) of the region of convergence of $\tilde{G}(z)$ in a dense fashion. We have thus shown that $\tilde{G}(z)$ has a dense, discrete set of zeros on the boundary of its region of convergence.

Sub-case (ii) [One of k and p odd, the other even] In this case $N = p$ and there is no simple way to see a general formula for $\sigma := \sum_{j=0}^{p-1} e^{i\pi k \frac{j^2}{p}}$, which is why, for example, the result by Gauss evaluating this sum when $k = 2$ is famous and its derivation here using Césaro methods in corollary 1 is interesting. At this stage we cannot therefore draw a general conclusion regarding $\tilde{G}(z)$ near $\nu_{k,p}$ in this case. All we can observe is that generically we would expect $\sigma \neq 0$ and that, in any instance where we could show this to be so (either by indirect means or simply by direct calculation), it would follow from equation 73 that $\tilde{G}(z)$ has a square-root branch-point singularity at $\nu_{k,p}$ on its boundary of convergence.

We will not pursue this further here, other than to note, at least, that there are certainly points arbitrarily far out along its boundary of convergence where $\tilde{G}(z)$ has a dense accumulation of such square-root branch-point singularities. Specifically, if we require p odd and set $k = 2lp + 2$, $l \in \mathbb{Z}_{>0}$, then we have that $e^{i\pi k \frac{j^2}{p}} = e^{2\pi i l j^2} e^{2\pi i \frac{j^2}{p}} = e^{2\pi i \frac{j^2}{p}}$ and we replicate Gauss' result, so that in this case

$$\sigma = \begin{cases} \sqrt{p} & , \ p \equiv 1 \ (mod\ 4) \\ \\ i\sqrt{p} & , \ p \equiv 3 \ (mod\ 4)\,. \end{cases} \tag{75}$$

Since $\sigma \neq 0$ at these points it follows that, as $p \to \infty$ we get square-root branch-point singularities accumulating densely on the point $\sqrt{-2li}$, and as $l \to \infty$ these points occur further and further out along the boundary of convergence of $\tilde{G}$ without bound.

Thus, without having a comprehensive resolution of the general case of $\nu_{k,p}$, we have at least proven enough to see that all along the boundary of the region of convergence of $\widetilde{G}(z)$ there are densely accumulating and densely interleaved discrete collections of zeros and square-root branch-point singularities.

3.3 A brief aside re discrete vs continuous Césaro-array methods

In our 2-d Césaro array analysis in this section we have adopted a discrete, rather than continuous, Césaro framework. Of course, since it ultimately gives a power series representation for the same well-defined underlying function ($f(\epsilon)$ or $\widetilde{G}(z)$ or $\ldots$), any such 2-d Césaro array analysis will give the same final power series, irrespective of whether we use a continuous or a discrete Césaro framework (or indeed any other generalised convergence scheme).

However, as we saw in the demonstration case of $f(\epsilon) = \frac{1}{\epsilon} = \sum_{j=0}^{\infty}(1-\epsilon)^j$ examined in [IV], we will often get variation regarding how things are split between component-1 and component-2 pieces in the course of such 2-d analysis, according to whether the analysis is conducted in a discrete or continuous Césaro framework.

In that example we saw that, if we used a discrete Césaro approach, all the horizontal sums at order ϵ^n, $n \in \mathbb{Z}_{\geq 0}$, were identically zero; with the component-2 piece which arises from recombining all the non-trivial eigensequence divergences at these levels then giving us precisely just the single expected remaining $\frac{1}{\epsilon}$ term. By contrast, in a continuous Césaro framework, the horizontal sums at each level ϵ^n, $n \in \mathbb{Z}_{\geq 0}$, were non-zero, giving us contributions $\frac{1}{2}\epsilon^0$, $\frac{1}{12}\epsilon^1$, $\frac{1}{24}\epsilon^2$, $\frac{19}{720}\epsilon^3$, $\ldots$. But the more complicated collection of divergent eigenfunctions at each level re-combined vertically to give a component-2 contribution of $-\frac{1}{\ln(1-\epsilon)}$. Since, for ϵ small we have $-\frac{1}{\ln(1-\epsilon)} = \frac{1}{\epsilon} - \frac{1}{2} - \frac{1}{12}\epsilon^1 - \frac{1}{24}\epsilon^2 - \frac{19}{720}\epsilon^3 - \ldots$, this component-2 piece cancelled the contributions of the component-1 horizontal sums and left us with the same single overall $\frac{1}{\epsilon}$ term. We thus ended up in the same place, albeit via a very different allocation of component-1 and component-2 contributions.

Now consider again the case of $f(\epsilon) = \sum_{j=0}^{\infty} \tilde{a}_j e^{-\pi j^2 \epsilon}$ where the coefficients $\{\tilde{a}_j\} \in \mathcal{S}_{[N]}$. We saw in case 1 that under the discrete Césaro framework the resulting horizontal sums are all rendered convergent by pure powers of P_D (P_D^{2n+1} for the horizontal sum at order ϵ^n). This was because $\{\tilde{a}_j\}$ may be taken as the base-sequence in a discrete Césaro-adapted scale, and it follows that there are no non-trivial divergent eigensequences at each height to re-combine, and so no overall component-2 contribution even to consider, leaving us with the power series expansion given in equation 52 in Result 2 for $f(\epsilon)$.

But in this case, if we change perspective and adopt instead a continuous Césaro framework, then the coefficient sequence $\{\tilde{a}_j\}$ - viewed as a step-function on $[0, \infty)$ - is still periodic with period-integral 0 ($\int_0^N \tilde{a}(x)\, dx = 0$) and so can equally well be taken as the base-function in a *continuous* Césaro-adapted scale. It thus follows, by exactly analogous reasoning to that employed in case 1, that in a continuous Césaro 2-d analysis each horizontal sum at order ϵ^n, $n \in \mathbb{Z}_{\geq 0}$, will likewise be rendered convergent by a pure power of the continuous Césaro operator P, namely P^{2n+1}, with no divergent eigenfunction pieces to sequester and re-combine vertically, and hence no component-2 contribution to consider either.

Thus in this case we see that, irrespective of whether we conduct our 2-d Césaro array analysis within a discrete or a continuous Césaro framework, we end up with zero component-2 contribution; and hence that our calculations of the individual degree-wise horizontal sums giving the coefficients $\tilde{A}_n$ of ϵ^n at each height n must agree, independent of this choice of framework, in order to end up with the same final power series for $f(\epsilon)$.

However, consider now the more general case where $\{a_j\}$ is period-N but not in $\mathcal{S}_{[N]}$ - so that we needed to take $\tilde{a}_j = a_j - \frac{\sigma}{N}$ where $\sigma := \sum_{j=0}^{N-1} a_j$. In this case, we see that this agreement, at the detailed level of horizontal sums and component-contributions, breaks down. Since $a_j = \tilde{a}_j + \frac{\sigma}{N}$, the detailed degree-wise calculation of each coefficient of ϵ^n will vary between the discrete and continuous Césaro frameworks exactly to the degree to which the evaluation of $\frac{\sigma}{N}\sum_{j=1}^{\infty} j^{2n}$ varies between these frameworks.

Now in [I] we showed that $\sum_{j=1}^{\infty} j^{2n} = 1$ for all $n \in \mathbb{Z}_{>0}$ within a discrete Césaro framework and this allowed us to derive the discrete Césaro sums quoted in equation 63 in corollary 2a and equation 68 in corollary 3. Had we instead been using a continuous Césaro framework, the corresponding results from [I] are that for all $n \in \mathbb{Z}_{>0}$, $\sum_{j=1}^{\infty} j^{2n} = \zeta(-2n) = 0$ via $q(P) = \left(\frac{2n+2}{2n+1}\right)\left(P - \frac{1}{2n+2}\right) \cdot P^{2n+1}$, and we would correspondingly have claimed instead that

$$\sum_{j=1}^{\infty} j^{2n} e^{2\pi i \frac{j^2}{p}} = 0 \quad \text{for all } n \in \mathbb{Z}_{>0} \tag{76}$$

and that

$$\sum_{j=1}^{\infty} j^{2n} e^{i\pi \frac{j^2}{p}} = 0 \quad \text{for all } n \in \mathbb{Z}_{>0} \tag{77}$$

within the continuous Césaro scheme.

Of course, as noted, the overall power series arising from full 2-d Césaro analysis is always the same regardless of which Césaro framework is adopted. Thus any differences which (as in these examples) occur in such horizontal sums as a consequence of which framework (discrete or continuous) we are using, must be exactly offset by corresponding differences in the component-2 piece which is created under each approach, owing to the different divergent eigensequences/eigenfunctions which accrue at each level under the two schemes and how these then re-combine vertically.

We omit any further calculations or detailed discussion here, but it is worth making clear this way in which the detailed breakdown between component-1 and component-2 pieces - and hence for example the content of equations like those just given, or equations 63 and 68 - is dependent on the particular choice of generalised convergence framework under which the 2-d array analysis is conducted.

4 Extending the results of section 3 for $\sum_{j=1}^{\infty}\tilde{a}_j j^{2n}$ where $\{\tilde{a}_j\}\in\mathcal{S}_{[N]}$ and $\tilde{a}_{N-j}=\tilde{a}_j$ for all j

Combining the results of cases 1 and 3 we showed in section 3 that if $\{\tilde{a}_j\}\in\mathcal{S}_{[N]}$ and $\tilde{a}_{N-j}=\tilde{a}_j$ for all j, then $f(\epsilon):=\sum_{j=0}^{\infty}\tilde{a}_j e^{-\pi j^2\epsilon}$ is given by the power series

$$f(\epsilon)=\frac{\tilde{a}_0}{2}+\sum_{n=1}^{\infty}\tilde{A}_n\epsilon^n+\mathcal{S}_0(\epsilon)\quad \text{as } \epsilon\to 0^+ \tag{78}$$

with no singular component. Here each $\tilde{A}_n$ is $(-1)^n\frac{\pi^n}{n!}$ times the degree-wise Césaro sum, via P_D^{2n+1}, of $\sum_{j=1}^{\infty}\tilde{a}_j j^{2n}$; and since these require only pure powers of P_D to be rendered classically convergent, so, in light of the observations just discussed in subsection 3.3, the Césaro evaluation of these sums is in fact independent of whether we use a discrete or continuous Césaro framework. For simplicity, however, we will continue to use the discrete framework in all that follows here.

Now in cases 4-6, where we could draw on alternative power series expansions derived in section 2, we saw that in fact we had $\tilde{A}_n=0$ for all $n\in\mathbb{Z}_{\geq 1}$. Thus, although we commented after both result 2 and corollary 5 about needing to be careful in this regard, we might be tempted to make the following reckless conjecture:

Conjecture 1: *If $\{\tilde{a}_j\}\in\mathcal{S}_{[N]}$ and satisfies $\tilde{a}_{N-j}=\tilde{a}_j$ for all j, then for any $n\in\mathbb{Z}_{\geq 1}$ we have that*

$$\sum_{j=1}^{\infty}\tilde{a}_j j^{2n}=0 \tag{79}$$

within a discrete Césaro framework, via the pure power P_D^{2n+1}; i.e. we have the following strong discrete Césaro limit for the p-sum sequence:

$$\sum_{j=1}^{k}\tilde{a}_j j^{2n}\overset{C_D}{\simeq}0 \tag{80}$$

since

$$P_D^{2n+1}\left[\sum_{j=1}^{\tilde{k}}\tilde{a}_j j^{2n}\right]_k\to 0\quad\text{classically as } k\to\infty\ . \tag{81}$$

In fact, we feel quite sure that this conjecture is true. However, since this paper is already too long, we content ourselves here with briefly outlining the broad steps which we believe would be involved in proving it in general; and with carrying out these steps to confirm the truth of equation 79 only in the first two cases $n = 1$ and $n = 2$. To facilitate this, let us fix some notation.

As previously, let $s^{(n)}$, $n \in \mathbb{Z}_{\geq 0}$, be the unique Césaro-adapted scale built from the base sequence $\{\tilde{a}_j\}_{j=-\infty}^{\infty}$, i.e. having $s^{(0)} = \tilde{a}$ as sequences. Then we have $\Delta s^{(n)} = s^{(n-1)}$ for all $n \in \mathbb{Z}_{\geq 1}$ and we have seen that $s^{(n)}$ is given in terms of $s^{(n-1)}$ by

$$s_j^{(n)} = \sum_{i=0}^{j} s_i^{(n-1)} - C^{(n)} \quad \text{for all } j \in \mathbb{Z} \tag{82}$$

where $C^{(n)}$ is the average value of the p-sum sequence of $s^{(n-1)}$ over its period, i.e.

$$C^{(n)} = \frac{1}{N} \sum_{j=0}^{N-1} \left\{ \sum_{i=0}^{j} s_i^{(n-1)} \right\} . \tag{83}$$

Note in passing that in the proof of result 1 we showed that, since $\{\tilde{a}_j\} \in \mathcal{S}_{[N]}$ and $\tilde{a}_{N-j} = \tilde{a}_j$, we also have $C^{(1)} = \frac{1}{2}\tilde{a}_0$. The basic steps for evaluating $\sum_{j=1}^{\infty} \tilde{a}_j j^{2n}$ within a discrete Césaro framework are then as follows.

Step 1 [Derive a formula for $\sum_{j=1}^{\infty} \tilde{a}_j j^l$ using summation by parts]: First we apply summation by parts to the p-sum sequence $\sum_{j=1}^{k} \tilde{a}_j j^l$ and then invoke lemmas 2 and 3 to obtain an expression for the discrete Césaro sum of $\sum_{j=1}^{\infty} \tilde{a}_j j^l$ in terms of the "constants of integration" $s_0^{(1)}$,$s_0^{(2)}$, …,$s_0^{(l+1)}$. For example, for $l = 0$, we have

$$\begin{aligned} \sum_{j=1}^{k} \tilde{a}_j &= \sum_{j=1}^{k} 1 \cdot (\Delta s^{(1)})_j = \left[s_k^{(1)} - s_0^{(1)} \right] - \sum_{j=1}^{k} s_{j-1}^{(1)} \cdot 0 \\ &= s_k^{(1)} - s_0^{(1)} \xrightarrow{C_D} -s_0^{(1)} \quad \text{as } k \to \infty \end{aligned}$$

while for $l = 1$, on applying summation by parts, re-arranging and invoking the result for $l = 0$ with $s^{(1)}$ in place of $\tilde{a}$, we have

$$\begin{aligned} \sum_{j=1}^{k} j\tilde{a}_j &= \sum_{j=1}^{k} j \cdot (\Delta s^{(1)})_j \;=\; \left[k s_k^{(1)} - 0 \cdot s_0^{(1)} \right] - \sum_{j=1}^{k} s_{j-1}^{(1)} \cdot (\Delta \tilde{j})_j \\ &= k s_k^{(1)} - \sum_{j=1}^{k} s_{j-1}^{(1)} \cdot 1 \;=\; k s_k^{(1)} - s_0^{(1)} + s_k^{(1)} - \sum_{j=1}^{k} s_j^{(1)} \cdot 1 \\ &= (k+1) s_k^{(1)} - s_k^{(2)} - s_0^{(1)} + s_0^{(2)} \xrightarrow{C_D} -s_0^{(1)} + s_0^{(2)} \quad \text{as } k \to \infty \end{aligned}$$

and for $l = 2$, on applying summation by parts, re-arranging and invoking the results for $l = 1$ and $l = 0$, we likewise have

$$\begin{aligned}
\sum_{j=1}^{k} j^2 \tilde{a}_j &= k^2 s_k^{(1)} - \sum_{j=1}^{k} s_{j-1}^{(1)} \cdot (2j-1) \\
&= k^2 s_k^{(1)} - s_0^{(1)} + (2k+1) s_k^{(1)} - \sum_{j=1}^{k} s_j^{(1)} \cdot (2j+1) \\
&= \{(k+1)^2 s_k^{(1)} - (2k+3) s_k^{(2)} + 2 s_k^{(3)}\} - \{s_0^{(1)} - 3 s_0^{(2)} + 2 s_0^{(3)}\} \\
&\xrightarrow{C_D} -s_0^{(1)} + 3 s_0^{(2)} - 2 s_0^{(3)} \quad \text{as } k \to \infty\ . \qquad (84)
\end{aligned}$$

Proceeding along the same lines as in these computations and letting T be the translation operator on sequences given by $T[\{a\}]_j := a_{j+1}$ (so that the discrete derivative operator Δ is closely related to $(T-1)$), it is not hard to see that in fact the general formula for $\sum_{j=1}^{\infty} j^l \tilde{a}_j$ is given by

$$\sum_{j=1}^{\infty} j^l \tilde{a}_j = \left\{ \begin{array}{c} -(T-1)^0[\{\tilde{j}^l\}]_1 s_0^{(1)} + (T-1)^1[\{\tilde{j}^l\}]_1 s_0^{(2)} \\ \\ -(T-1)^2[\{\tilde{j}^l\}]_1 s_0^{(3)} + \ldots + (-1)^{l+1}(T-1)^l[\{\tilde{j}^l\}]_1 s_0^{(l+1)} \end{array} \right\} \qquad (85)$$

and thus, for example, for $l = 4$ we have the discrete Césaro sum

$$\begin{aligned}
\sum_{j=1}^{\infty} j^4 \tilde{a}_j &= \left\{ \begin{array}{c} -s_0^{(1)} + (2^4 - 1) s_0^{(2)} - (3^4 - 2 \cdot 2^4 + 1) s_0^{(3)} \\ +(4^4 - 3 \cdot 3^4 + 3 \cdot 2^4 - 1) s_0^{(4)} \\ -(5^4 - 4 \cdot 4^4 + 6 \cdot 3^4 - 4 \cdot 2^4 + 1) s_0^{(5)} \end{array} \right\} \\
&= -s_0^{(1)} + 15 s_0^{(2)} - 50 s_0^{(3)} + 60 s_0^{(4)} - 24 s_0^{(5)}\ . \qquad (86)
\end{aligned}$$

Step 2 [Translate this formula into an expression in the constants $C^{(i)}$]: In light of equation 82 it is immediate that

$$\begin{aligned}
s_0^{(1)} &= \tilde{a}_0 - C^{(1)} \quad \text{and} \\
s_0^{(2)} &= \tilde{a}_0 - C^{(1)} - C^{(2)} \quad \text{and} \\
s_0^{(3)} &= \tilde{a}_0 - C^{(1)} - C^{(2)} - C^{(3)} \quad \text{and} \\
&\vdots \\
s_0^{(m)} &= \tilde{a}_0 - C^{(1)} - \ldots - C^{(m)}\ . \qquad (87)
\end{aligned}$$

Substituting this into equation 85 we can recast our formula for $\sum_{j=1}^{\infty} j^l \tilde{a}_j$ in terms of the constants $C^{(i)}$, $1 \le i \le l+1$. For example, for our two demonstration cases of $\sum_{j=1}^{\infty} j^2 \tilde{a}_j$ and $\sum_{j=1}^{\infty} j^4 \tilde{a}_j$, equations 84 and 86 become

$$\sum_{j=1}^{\infty} j^2 \tilde{a}_j = -C^{(2)} + 2C^{(3)} \qquad (88)$$

and

$$\sum_{j=1}^{\infty} j^4 \tilde{a}_j = -C^{(2)} + 14C^{(3)} - 36C^{(4)} + 24C^{(5)} \ . \tag{89}$$

Step 3 [Derive an expression for $C^{(n)}$ in terms of $C^{(i)}$, $1 \leq i < n$, together with an iterated sum of $\tilde{a}_j$]: From equations 82 and 83 it follows that

$$C^{(1)} = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{j=0}^{k} \tilde{a}_j$$

and

$$\begin{aligned}
C^{(2)} &= \frac{1}{N} \sum_{k=0}^{N-1} \sum_{j=0}^{k} s_j^{(1)} = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{j=0}^{k} \left\{ \sum_{i=0}^{j} \tilde{a}_i - C^{(1)} \right\} \\
&= \frac{1}{N} \left\{ \sum_{k=0}^{N-1} \sum_{j=0}^{k} \sum_{i=0}^{j} \tilde{a}_i - \sum_{k=0}^{N-1} \sum_{j=0}^{k} C^{(1)} \right\} \\
&= \frac{1}{N} \left\{ \sum_{k=0}^{N-1} \sum_{j=0}^{k} \sum_{i=0}^{j} \tilde{a}_i - \binom{N+1}{2} C^{(1)} \right\}
\end{aligned}$$

and

$$\begin{aligned}
C^{(3)} &= \frac{1}{N} \sum_{k=0}^{N-1} \sum_{j=0}^{k} s_j^{(2)} = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{j=0}^{k} \left\{ \sum_{i=0}^{j} s_i^{(1)} - C^{(2)} \right\} \\
&= \frac{1}{N} \left\{ \sum_{k=0}^{N-1} \sum_{j=0}^{k} \sum_{i=0}^{j} \left\{ \sum_{n=0}^{i} \tilde{a}_n - C^{(1)} \right\} - \binom{N+1}{2} C^{(2)} \right\} \\
&= \frac{1}{N} \left\{ \sum_{k=0}^{N-1} \sum_{j=0}^{k} \sum_{i=0}^{j} \sum_{n=0}^{i} \tilde{a}_n - \binom{N+2}{3} C^{(1)} - \binom{N+1}{2} C^{(2)} \right\}
\end{aligned}$$

and in general

$$C^{(n)} = \frac{1}{N} \left\{ \begin{array}{c} \sum_{k_n=0}^{N-1} \sum_{k_{n-1}=0}^{k_n} \cdots \sum_{k_0=0}^{k_1} \tilde{a}_{k_0} - \binom{N+n-1}{n} C^{(1)} \\ -\binom{N+n-2}{n-1} C^{(2)} - \ldots - \binom{N+1}{2} C^{(n-1)} \end{array} \right\} . \tag{90}$$

Step 4 [Find an expression for the iterated sum of $\tilde{a}_{k_0}$ in equation 90 instead in terms of finite single-period sums $\sum_{j=0}^{N-1} \tilde{a}_j j^l$, $1 \leq l \leq n$]: The first sub-step (sub-step 4a) in simplifying the iterated sum in equation 90 is to show that, in fact, for any $L \in \mathbb{Z}_{>0}$, such an iterated sum can be collapsed to a single sum as follows:

$$\sum_{k_n=0}^{L} \sum_{k_{n-1}=0}^{k_n} \cdots \sum_{k_0=0}^{k_1} \tilde{a}_{k_0} = \sum_{j=0}^{L} \binom{L+n-j}{n} \tilde{a}_j \ . \tag{91}$$

The proof of this is by induction on n but we omit any further details here. Next (sub-step 4b), if for any $l \in \mathbb{Z}_{\geq 1}$ we let $\sigma^{(l)}$ denote the period-sum

$$\sigma^{(l)} := \sum_{j=1}^{N-1} \tilde{a}_j \cdot j^l \tag{92}$$

then, setting $L = N-1$ in equation 91 and expanding $\binom{N+(n-1)-j}{n}$ as a degree n polynomial in j, namely

$$\binom{N+(n-1)-j}{n} := d_0^{(n)} + d_1^{(n)} j + \ldots + d_n^{(n)} j^n \tag{93}$$

it follows that the iterated sum in equation 90 can be expressed in terms of the $\sigma^{(l)}$ as

$$\sum_{k_n=0}^{N-1} \sum_{k_{n-1}=0}^{k_n} \cdots \sum_{k_0=0}^{k_1} \tilde{a}_{k_0} = d_1^{(n)} \sigma^{(1)} + \ldots + d_n^{(n)} \sigma^{(n)} \tag{94}$$

and this completes the task in this step. For example, consider the cases $n = 2, 3, 4$ and 5 which we will need for our two demonstration calculations; for $n = 2$ we have

$$\sum_{k_2=0}^{N-1} \sum_{k_1=0}^{k_2} \sum_{k_0=0}^{k_1} \tilde{a}_{k_0} \; = \; \frac{1}{2}\sigma^{(2)} - \frac{1}{2}(2N+1)\sigma^{(1)} \tag{95}$$

and, for $n = 3$, we have

$$\sum_{k_3=0}^{N-1} \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} \sum_{k_0=0}^{k_1} \tilde{a}_{k_0} \; = \; -\frac{1}{6}\sigma^{(3)} + \frac{1}{2}(N+1)\sigma^{(2)} - \frac{1}{6}(3N^2+6N+2)\sigma^{(1)} \tag{96}$$

and for $n = 4$ we have

$$\sum_{k_4=0}^{N-1} \sum_{k_3=0}^{k_4} \cdots \sum_{k_0=0}^{k_1} \tilde{a}_{k_0} \; = \; \left\{ \begin{array}{c} \frac{1}{24}\sigma^{(4)} - \frac{1}{12}(2N+3)\sigma^{(3)} \\ \\ +\frac{1}{24}(6N^2+18N+11)\sigma^{(2)} \\ \\ -\frac{1}{24}(4N^3+18N^2+22N+6)\sigma^{(1)} \end{array} \right\} \tag{97}$$

and for $n = 5$ we have

$$\sum_{k_5=0}^{N-1} \sum_{k_4=0}^{k_5} \cdots \sum_{k_0=0}^{k_1} \tilde{a}_{k_0} \; = \; \left\{ \begin{array}{c} -\frac{1}{120}\sigma^{(5)} + \frac{1}{24}(N+2)\sigma^{(4)} \\ \\ -\frac{1}{24}(2N^2+8N+7)\sigma^{(3)} \\ \\ +\frac{1}{24}(2N^3+12N^2+21N+10)\sigma^{(2)} \\ \\ -\frac{1}{120}(5N^4+40N^3+105N^2+100N+24)\sigma^{(1)} \end{array} \right\} \tag{98}$$

So far, for steps 1-4, we have considered $\sum_{j=1}^{\infty} \tilde{a}_j j^l$ in general for arbitrary $l \in \mathbb{Z}_{\geq 1}$, not just $l = 2n$ even; and we have only really used the fact that $\{\tilde{a}_j\} \in \mathcal{S}_{[N]}$. We have not so far used the additional symmetry condition that $\tilde{a}_{N-j} = \tilde{a}_j$ other than for making the passing observation that then $C^{(1)} = \frac{\tilde{a}_0}{2}$. In the next step, however, this condition becomes critical and is central to why we can ultimately calculate $\sum_{j=1}^{\infty} \tilde{a}_j j^l$ when $l = 2n$ is even.

Step 5 [For $l = 2m+1$ odd, find an expression for $\sigma^{(l)}$ in terms of $\sigma^{(i)}$, $1 \leq i < l$]: Since $\tilde{a}_{N-j} = \tilde{a}_j$ it follows, on noting that $\sigma^{(0)} = -\tilde{a}_0$, that

$$\begin{aligned}
\sigma^{(l)} &= \sum_{j=1}^{N-1} \tilde{a}_j \cdot j^l \;=\; \frac{1}{2}\left\{ \sum_{j=1}^{N-1} \tilde{a}_j \cdot j^l + \sum_{j=1}^{N-1} \tilde{a}_{N-j} \cdot (N-j)^l \right\} \\
&= \frac{1}{2} \sum_{j=1}^{N-1} \tilde{a}_j \cdot \{ j^l + (N-j)^l \} \\
&= \frac{1}{2} \sum_{j=1}^{N-1} \tilde{a}_j \cdot \left\{ j^l + \left[\begin{array}{c} N^l - \binom{l}{1} N^{l-1} j + \binom{l}{2} N^{l-2} j^2 \\ - \ldots + (-1)^{l-1} \binom{l}{l-1} N j^{l-1} + (-1)^l j^l \end{array} \right] \right\} \\
&= \frac{1}{2} \left\{ \begin{array}{c} -N^l \tilde{a}_0 - \binom{l}{1} N^{l-1} \sigma^{(1)} + \binom{l}{2} N^{l-2} \sigma^{(2)} - \ldots \\ +(-1)^{l-1} \binom{l}{l-1} N \sigma^{(l-1)} + [1 + (-1)^l] \sigma^{(l)} \end{array} \right\} \qquad (99)
\end{aligned}$$

When $l = 2m+1$ is odd the $\sigma^{(l)}$ terms on the RHS cancel and we are left with an expression for $\sigma^{(l)}$ in terms of $\sigma^{(i)}$, $1 \leq i < l$, as desired. For example, in the cases relevant to our demonstration calculations, we have:

$$\sigma^{(1)} = -\frac{1}{2} N \tilde{a}_0 \qquad (100)$$

and

$$\sigma^{(3)} = -\frac{1}{2} N^3 \tilde{a}_0 - \frac{3}{2} N^2 \sigma^{(1)} + \frac{3}{2} N \sigma^{(2)} \qquad (101)$$

and

$$\sigma^{(5)} = -\frac{1}{2} N^5 \tilde{a}_0 - \frac{5}{2} N^4 \sigma^{(1)} + 5 N^3 \sigma^{(2)} - 5 N^2 \sigma^{(3)} + \frac{5}{2} N \sigma^{(4)} \; . \qquad (102)$$

Step 6 [Derive an expression for the constants $C^{(l)}$ purely in terms of $\sigma^{(i)}$, $1 \leq i \leq l$]: Combining steps 3 and 4 we can express the constants $C^{(l)}$ purely in terms of the quantities $\sigma^{(i)}$, $1 \leq i \leq l$. For example, for our demonstration cases, using equations 95-98 from step 4 iteratively in our formulae for $C^{(l)}$ from step 3, and noting that $NC^{(1)} = -\sigma^{(1)}$, we have that

$$NC^{(2)} \;=\; \frac{1}{2}\sigma^{(2)} - \frac{1}{2}(2N+1)\sigma^{(1)} + \frac{1}{2}(N+1)\sigma^{(1)} \;=\; \frac{1}{2}\sigma^{(2)} - \frac{1}{2} N \sigma^{(1)} \qquad (103)$$

and

$$\begin{aligned} NC^{(3)} &= \left\{ \begin{array}{c} -\tfrac{1}{6}\sigma^{(3)} + \tfrac{1}{2}(N+1)\sigma^{(2)} - \tfrac{1}{6}(3N^2+6N+2)\sigma^{(1)} \\ \\ +\tfrac{1}{6}(N^2+3N+2)\sigma^{(1)} - \tfrac{1}{2}(N+1)\left\{\tfrac{1}{2}\sigma^{(2)} - \tfrac{1}{2}N\sigma^{(1)}\right\} \end{array} \right\} \\ &= -\frac{1}{6}\sigma^{(3)} + \frac{1}{4}(N+1)\sigma^{(2)} - \frac{1}{12}(N^2+3N)\sigma^{(1)} \qquad (104) \end{aligned}$$

and, in the same fashion, after slightly lengthier derivations,

$$NC^{(4)} = \frac{1}{24}\sigma^{(4)} - \frac{1}{12}(N+2)\sigma^{(3)} + \frac{1}{24}(N^2+6N+4)\sigma^{(2)} - \frac{1}{12}(N^2+2N)\sigma^{(1)} \qquad (105)$$

and

$$NC^{(5)} = \left\{ \begin{array}{c} -\tfrac{1}{120}\sigma^{(5)} + \tfrac{1}{48}(N+3)\sigma^{(4)} - \tfrac{1}{72}(N^2+9N+11)\sigma^{(3)} \\ \\ +\tfrac{1}{48}(3N^2+11N+6)\sigma^{(2)} + \tfrac{1}{720}(N^4-55N^2-90N)\sigma^{(1)} \end{array} \right\} \qquad (106)$$

Step 7 [Perform the calculation of $\sum_{j=1}^{\infty} \tilde{a}_j j^{2n}$]: Using these results from step 6, the expression for $\sum_{j=1}^{\infty} \tilde{a}_j j^{2n}$ in step 2 can thus be re-expressed in terms of $\sigma^{(m)}$ only, and the identities derived in step 5 can then be invoked to simplify and calculate these expressions. For example, in our two demonstration calculations:

[Demonstration calculation 1 (n=1) for $\sum_{j=1}^{\infty} \tilde{a}_j j^2$]: We have

$$\begin{aligned} N \cdot \sum_{j=1}^{\infty} \tilde{a}_j j^2 &= N \cdot \{-C^{(2)} + 2C^{(3)}\} \\ &= \left\{ \begin{array}{c} -\tfrac{1}{3} \cdot \left\{ \begin{array}{c} -\tfrac{1}{2}N^3\tilde{a}_0 - \tfrac{3}{2}N^2\sigma^{(1)} \\ +\tfrac{3}{2}N\sigma^{(2)} \end{array} \right\} + \tfrac{1}{2}(N+1)\sigma^{(2)} \\ \\ -\tfrac{1}{6}(N^2+3N)\sigma^{(1)} - \left[\tfrac{1}{2}\sigma^{(2)} - \tfrac{1}{2}N\sigma^{(1)}\right] \end{array} \right\} \\ &= \frac{1}{6}N^3\tilde{a}_0 + \frac{1}{3}N^2 \cdot \left(-\frac{1}{2}N\tilde{a}_0\right) = 0 \end{aligned}$$

and thus

$$\sum_{j=1}^{\infty} \tilde{a}_j j^2 = 0 \qquad (107)$$

in a discrete (and also continuous) Césaro sense, as claimed. Likewise:

[Demonstration calculation 2 (n=2) for $\sum_{j=1}^{\infty}\tilde{a}_j j^4$]: We have

$$
\begin{aligned}
N\cdot\sum_{j=1}^{\infty}\tilde{a}_j j^4 &= N\cdot\{-C^{(2)}+14C^{(3)}-36C^{(4)}+24C^{(5)}\}\\
&= \left\{\begin{array}{c}
\left\{\begin{array}{c}
-\tfrac{1}{5}\sigma^{(5)}+\tfrac{1}{2}(N+3)\sigma^{(4)}-\tfrac{1}{3}(N^2+9N+11)\sigma^{(3)}\\
+\tfrac{1}{2}(3N^2+11N+6)\sigma^{(2)}\\
+\tfrac{1}{30}(N^4-55N^2-90N)\sigma^{(1)}
\end{array}\right\}\\
-\left\{\begin{array}{c}
\tfrac{3}{2}\sigma^{(4)}-3(N+2)\sigma^{(3)}+\tfrac{3}{2}(N^2+6N+4)\sigma^{(2)}\\
-3(N^2+2N)\sigma^{(1)}
\end{array}\right\}\\
+\left\{-\tfrac{7}{3}\sigma^{(3)}+\tfrac{7}{2}(N+1)\sigma^{(2)}-\tfrac{7}{6}(N^2+3N)\sigma^{(1)}\right\}\\
-\left\{\tfrac{1}{2}\sigma^{(2)}-\tfrac{1}{2}N\sigma^{(1)}\right\}
\end{array}\right\}\\
&= \left\{\begin{array}{c}
-\tfrac{1}{5}\cdot\left\{\begin{array}{c}
-\tfrac{1}{2}N^5\tilde{a}_0-\tfrac{5}{2}N^4\sigma^{(1)}+5N^3\sigma^{(2)}\\
-5N^2\sigma^{(3)}+\tfrac{5}{2}N\sigma^{(4)}
\end{array}\right\}\\
+\tfrac{1}{2}N\sigma^{(4)}-\tfrac{1}{3}N^2\sigma^{(3)}+\tfrac{1}{30}N^4\sigma^{(1)}
\end{array}\right\}\\
&= \left\{\begin{array}{c}
\tfrac{2}{3}N^2\cdot\left\{-\tfrac{1}{2}N^3\tilde{a}_0-\tfrac{3}{2}N^2\sigma^{(1)}+\tfrac{3}{2}N\sigma^{(2)}\right\}\\
-N^3\sigma^{(2)}+\tfrac{8}{15}N^4\sigma^{(1)}+\tfrac{1}{10}N^5\tilde{a}_0
\end{array}\right\}\\
&= -\frac{7}{15}N^4\sigma^{(1)}-\frac{7}{30}N^5\tilde{a}_0\\
&= 0
\end{aligned}
$$

and thus we have

$$\sum_{j=1}^{\infty}\tilde{a}_j j^4 = 0 \tag{108}$$

in a discrete (and also continuous) Césaro sense, also as claimed.

Final Comments: (i) In the above we have proven conjecture 1 for the two simplest cases $n = 1$ and $n = 2$. We have, in fact, also verified it for $n = 3$ by extending the results in each of the sequence of steps above, but we omit the calculations here since they are considerably longer even than the rather involved computations just outlined for $n = 1$ and $n = 2$.

As noted before, these computations suffice to make us quite certain that the conjecture is true in general for all $n \in \mathbb{Z}_{\geq 1}$. At the same time, however, it is clear that proving it in general will require further insights, either to bring greater order and tractability to the results derived in each of the preliminary steps 1-6, or else to find a different path altogether to the result. Brute force assault by direct computation is already stretched to the limit in the case of $n = 3$, and trying to proceed further in the same manner seems certain to end with the main force bogged down in attritional stalemate. We thus urge members of some more specialised forces to take up the task of finding a breakthrough. In particular we invite combinatorialist sappers more capable than ourselves to bring their skills to bear[5] on undermining the conjecture's defences and generalizing the working above into a full proof.

Some intriguing related combinatorial conjectures using formal symbols: As an inducement for combinatorialists and algebraists to take on this task, we note that many of the results obtained in steps 1-6 above seem to us already to be combinatorially interesting and to hint at hidden formal territory worth exploring in its own right[6]. For example, we conclude this comment by noting two ways of formally recasting the combinatorial results of step 5, both of which we find fascinating and which we believe are well worth investigating in greater detail (albeit that we have so far been unable to see how to harness these reformulations in service of a full proof).

In equations 100 - 102 in step 5 we derived expressions for $\sigma^{(l)} := \sum_{j=1}^{N-1} \tilde{a}_j j^l$, l odd, in terms of lower $\sigma^{(i)}$, $1 \leq i < l$ and $\tilde{a}_0$. If we extend the definition of $\sigma^{(i)}$ to $i = 0$ to get $\sigma^{(0)} := \sum_{j=1}^{N-1} \tilde{a}_j = -\tilde{a}_0$, then these three equations can all be re-expressed in much simpler (Umbral-style) formal terms as

$$\sigma^n + (\sigma - N)^n = 0 \quad \text{for } n = 1, 3, 5 \tag{109}$$

where we are defining powers of the formal symbol σ by $\sigma^i = \sigma^{(i)}$ in the usual manner (see e.g. [III]).

A little further working in turn shows that for $n = 2, 4$ or 6 we correspondingly have

$$\sigma^n - (\sigma - N)^n = 0 \tag{110}$$

since in each case we can re-write the expressions on the LHS in terms of the expressions for $n = 1, 3$ and 5 in equation 109.

We thus appear to have in general that

$$\sigma^n = (-1)^n \cdot (\sigma - N)^n \quad \text{for all } n \in \mathbb{Z}_{\geq 0} \tag{111}$$

whenever we start with a symmetric and strongly-periodic period-N sequence $\{\tilde{a}_j\}$, and this seems to us to be a fascinating formal conjecture worthy of proof and further study in its own right[7].

At the same time, another way of recasting these equations arises from recalling that the $\sigma^{(l)} := \sum_{j=1}^{N-1} \tilde{a}_j j^l$ are obtained by combining the $\tilde{a}_j$ with pure

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[5] a la Hill 60 in WW1

[6] See e.g. equations 85 and 89 in steps 1 and 2, and several others in later steps

[7] For example, if true, would equation 111 then extend from $n \in \mathbb{Z}_{\geq 0}$ to arbitrary $s \in \mathbb{C}$?

powers j^l. What happens if, in line with a philosophy we have often followed for being better adapted to discrete Césaro computation, we combine the $\tilde{a}_j$ instead with the alternative degree-l polynomials given by the binomial coefficients $\binom{(N-1)+l-j}{l}$?

Well, if we set $\hat{\sigma}^{(l)} := \sum_{j=1}^{N-1} \binom{(N-1)+l-j}{l} \tilde{a}_j$ and write $\hat{\sigma}^l = \hat{\sigma}^{(l)}$ formally in the usual way, then it turns out we can show that

$$\hat{\sigma}^3 = \frac{1}{2}\binom{N+2}{1}\hat{\sigma}^2 - \frac{1}{6}\binom{N+2}{2}\hat{\sigma}^1 \tag{112}$$

and

$$\hat{\sigma}^5 = \frac{1}{2}\binom{N+4}{1}\hat{\sigma}^4 - \frac{1}{6}\binom{N+4}{2}\hat{\sigma}^3 + \frac{1}{30}\binom{N+4}{4}\hat{\sigma}^1 \tag{113}$$

and

$$\hat{\sigma}^7 = \frac{1}{2}\binom{N+6}{1}\hat{\sigma}^6 - \frac{1}{6}\binom{N+6}{2}\hat{\sigma}^5 + \frac{1}{30}\binom{N+6}{4}\hat{\sigma}^3 - \frac{1}{42}\binom{N+6}{6}\hat{\sigma}^1 . \tag{114}$$

If we introduce a new (and non-symmetric) binomial expansion, one adapted to period-N sequences, by defining

$$\begin{aligned} (a+b)^{[n]} &:= a^n b^0 + \binom{(N-1)+n}{1} a^{n-1} b^1 + \ldots + \binom{(N-1)+n}{n} a^0 b^n \\ &= \sum_{j=0}^{n} \binom{(N-1)+n}{j} a^{n-j} b^j \end{aligned} \tag{115}$$

then these equations can all be re-expressed in simpler formal terms as

$$(\hat{\sigma} + B)^{[n]} = 0 \quad \text{for } n = 3, 5, 7 \tag{116}$$

where B is the formal symbol for the Bernoulli numbers (see [III]) satisfying $B^i = B_i$[8].

This leads naturally to the conjecture that $(\hat{\sigma} + B)^{[n]} = 0$ for all $n > 1$ odd. As with our first re-formulation, this seems to us to be a very intriguing formal relationship in and of itself, hinting as it does at some sort of generalisation of binomial relationships adapted to periodic sequences and yet still connecting back to the Bernoulli numbers or, equivalently, to values of ζ. Proving it in general, as well as extending it to cover n even and to the case of $n = 1$ (both of which may require a slight modification to the definition of $\hat{\sigma}^{(l)}$), seem to us to be interesting challenges in their own right.

(ii) If we could prove conjecture 1 in general then, for any symmetric and strongly-periodic, period-N sequence $\{\tilde{a}_j\}$, we would have that $\sum_{j=1}^{\infty} \tilde{a}_j j^l = 0$

---

[8] Recall that B is given by the formal relationships $B^0 = B_0 = 1$ and $B^n = (B+1)^n$ for all $n \in \mathbb{Z}_{\geq 2}$, so that $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$ and so on, while $B_{2n+1} = 0$ for all $n \in \mathbb{Z}_{\geq 1}$

for any $l \in \mathbb{Z}_{>0}$ even. This would naturally raise the question of whether we can also evaluate $\sum_{j=1}^{\infty} \tilde{a}_j j^l$ for $l \in \mathbb{Z}_{>0}$ odd? In particular it would be interesting to understand whether $\sum_{j=1}^{\infty} \tilde{a}_j j^l$ being zero (conjecturally) for all l even is related to the fact that $\zeta(-2) = \zeta(-4) = \zeta(-6) = \ldots = 0$, and in turn, whether the value of $\sum_{j=1}^{\infty} \tilde{a}_j j^l$ for l odd is related to the value of $\zeta(-l)$. Another possible issue remaining to explore is whether the symmetry condition requiring that $\tilde{a}_{N-j} = \tilde{a}_j$, which we have imposed in this conjecture and in a number of earlier results, can be relaxed or amended in any way.

(iii) Finally, note that if we assume conjecture 1 is true then result 4 from case 3 in section 3 would give rise to the following result:

Result 5 (conditional on Conjecture 1): *If* $\{\tilde{a}_j\}_{j=-\infty}^{\infty}$ *is in* $\mathcal{S}_{[N]}$ *and satisfies* $\tilde{a}_{N-j} = \tilde{a}_j$ *for all* j *then* $f(\epsilon) := \frac{\tilde{a}_0}{2} + \sum_{j=1}^{\infty} \tilde{a}_j e^{-\pi j^2 \epsilon}$ *is Schwartzian near* 0, *i.e.*

$$f(\epsilon) = \mathcal{S}_0(\epsilon) \quad \text{as } \epsilon \to 0^+ \,. \tag{117}$$

This would then supply us with a mapping from such sequences to Schwartzian functions and a way of engineering Schwartzian functions as linear combinations of such sequence-generated Schwartzian functions. The question of whether an arbitrary Schwartzian function could in fact be viewed as a finite (or infinite) linear combination of such sequence-generated Schwartzian functions (of different periods) then becomes a natural next question to explore; as does the question of whether there is an easy calculus relating operations on generator sequences to operations on their associated Schwartzian functions; as does the question of whether we can decompose an arbitrary sequence into a superposition of such strongly-periodic, symmetric sequences (at least for some subspace of sequences) and, if so, what this might imply for the function associated with the general sequence; and so on.

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