

# Césaro Arrays III

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## Abstract

In [IV] and [V] we have defined 2-d Césaro array methodology and demonstrated its utility in deriving interesting and non-trivial results regarding asymptotic analysis and exponential sums; and in the analysis of the singularity structure of functions of a complex variable, at least modulo Schwartzian functions. The focus, however, has been practical and applied, and a rigorous justification of the methodology, and identification of its domain of applicability, has not been supplied. In this paper we begin the task of filling this gap. We refine the validity question into a precise mathematical statement and validate it in the three cases considered in [IV]. This working in turn suggests an approach to validating it in general, based on combining the precise error estimate in the Euler-McLaurin sum formula with the Fourier series decomposition of the functions  $\check{q}_\rho(\alpha)$  (and in particular of  $\check{q}_0(\alpha)$ ) derived in [III]. By proceeding along these lines, we are able to obtain a result validating the method in general under certain conditions, although much remains open regarding how much these conditions may be further loosened or amended.

## 1 Introduction

In [IV] and [V] we have introduced the method of 2-d Césaro arrays. Where a function is defined by an infinite sum of more familiar functions with known power series expansions<sup>1</sup>, the Césaro array methodology invokes generalised Césaro convergence methods (either continuous or discrete) to facilitate the reversal of the order of summation and thereby the calculation of a power series expansion for the sum function, at least modulo Schwartzian functions.

In these papers we focussed entirely on its utility as a method of asymptotic analysis. In [IV] we gave a number of examples of its successful use along these lines, including in particular the case of the famous function  $\check{G}(u) := \frac{1}{2} + \sum_{j=1}^{\infty} e^{-\pi j^2 u^2}$  and its behaviour near  $u = 0$ . In [V] we then extended the analysis for this latter example in several new directions - we derived the remarkable boundary singularity structure of  $\check{G}$ ; we deduced from this a number of well-known results regarding finite exponential sums and in turn extended these

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<sup>1</sup>be they Taylor series or asymptotic series; near 0,  $\infty$  or elsewhere

results in several new directions; and we considered the general case of functions defined by series with periodic coefficients to obtain new results and conjectures regarding Césaro sums, exponential sums and Schwartzian functions.

These calculations provide convincing evidence of the validity and usefulness of these Césaro-array methods, but it is nonetheless true that a rigorous validation of the method in general, as articulated at the end of section 2.1 in [IV], has not been provided. Just as with other tools used to facilitate reversal of order of summation and hence successful asymptotic analysis - for example Fubini-type theorems or methods involving the dominated convergence theorem - it is important to supply such a validation. The purpose of this paper is to begin this task and to derive an initial theorem in this direction - providing general conditions under which Césaro-array methodology holds true. We will also then discuss briefly how the general conditions stipulated in this theorem might be relaxed or amended. Since the method as outlined in [IV, section 2.1] was for continuous Césaro methods, we will focus here on this context, albeit that we have seen that the methodology works equally well in a discrete Césaro setting.

In section 2.1 we begin by specifying an initial class of cases to which 2-d Césaro-array methodology might be applied, and we deduce a precise mathematical claim encapsulating the methodology's validity for this class. The methodology as stated in [IV] is so general that the class we consider does not encompass all possible scenarios of application, but it is nonetheless a broad class and, in particular, one which covers all the examples considered in [IV] and [V].

In section 2.2 we then review each of our three key examples from [IV] from the point of view of this precise validity requirement. As we do so, we note which cases give an *exact* final power series from our Césaro array analysis, and which only give agreement modulo Schwartzian functions. In the first example, that of  $f(u) := \sum_{j=1}^{\infty} e^{-ju}$ , where we know that the resulting power series is exact, we are able to confirm this directly by algebraic manipulation of the formal expression on the LHS in our validity condition. Such algebra cannot, however, be readily applied to our other two examples - namely  $\tilde{G}(u) := \frac{1}{2} + \sum_{j=1}^{\infty} e^{-\pi j^2 u^2}$  and  $f(u) := -\frac{1}{u} + \sum_{j=1}^{\infty} \frac{1}{(ju+1)^2}$ .

For these we instead use a result from [III] to obtain the derivative of the formal expression on the LHS in our validity condition and we integrate this to reconstruct it as a difference between an integral and a sum. To get a handle on this, we then apply the Euler-McLaurin sum formula - but a version of it with a precise formulation of the error estimate in terms of an integral involving the function  $\check{q}_0(\alpha)$ .<sup>2</sup> By invoking the Fourier series decomposition for  $\check{q}_0(\alpha)$  derived in [III]<sup>3</sup>, we are then able to perform the calculation of this integral in both the remaining cases. This allows us to confirm the correctness of our validity

<sup>2</sup>Thus far, in papers [I]-[V], we have quoted and used only a version of the Euler-McLaurin sum formula which does not specify this error term precisely

<sup>3</sup>Or obtainable in many other places in the literature concerning Bernoulli polynomials (see e.g. [6, Chapter 5])

condition again in both instances - albeit that in these cases our final expression is not identically zero, but only in  $\mathcal{S}_0(u)$ .

This latter approach - integrating the derivative of our formal expression, applying the Euler-McLaurin sum formula, and then precisely estimating the resulting integral using the Fourier decomposition of  $\check{q}_0(\alpha)$  - provides us with the way to attack the question of the validity of the Césaro-array methodology in *general*. To this end, in section 2.3 we start by imposing a further condition on the class of cases we are considering. Our aim is to simplify any convergence issues involved in our error integral calculations, while not being unduly restrictive. Under this condition we establish general results for the validity of Césaro-array analysis, first in the case of an even building-block function, then in the case of an odd building-block function (where the argument remains a little loose), and hence finally for arbitrary such building-block functions.

The extra condition imposed in section 2.3 still encompasses the first two of our three examples from [IV], and certainly covers a wide variety of other interesting cases, but it does not cover the last case of  $f(u) := -\frac{1}{u} + \sum_{j=1}^{\infty} \frac{1}{(ju+1)^2}$ . In section 2.4 we thus begin the process of considering how our proofs from section 2.3 might be adapted to handle this example and, more generally, how they might be amended to handle a broader relaxation of the condition imposed in section 2.3.

It is important to note, however, that since the general setting for the application of Césaro-array methodology is almost limitlessly broad, there remains enormous scope for further general results regarding its validity, beyond the initial result and discussion provided in this paper. We touch on this in brief concluding remarks in section 3.

## 2 The validity of Césaro array methodology in general

The initial class of cases for which we will try to establish the validity of Césaro array methodology is as follows. Let  $\phi(z)$  be a smooth function which decays to 0 as  $z \rightarrow \infty$  sufficiently fast to ensure that  $\phi$  is integrable on  $[0, \infty)$  and that  $\sum_{j=1}^{\infty} \phi(ju)$  is classically convergent and finite for any given  $u \in \mathbb{R}_{>0}$ , i.e. we have

$$\phi(z) \rightarrow 0 \quad \text{as } z \rightarrow \infty \tag{1}$$

and

$$\int_0^{\infty} \phi(z) dz = L \quad \text{for some finite } L \tag{2}$$

and

$$\sum_{j=1}^{\infty} \phi(ju) \quad \text{is well-defined and finite for any } u \in \mathbb{R}_{>0}. \tag{3}$$

Writing  $f(u) := \sum_{j=1}^{\infty} \phi(ju)$  we wish to understand whether Césaro-array methodology can always be applied to deduce a power series for  $f(u)$  as  $u \rightarrow 0^+$ .<sup>4</sup> Note that our three key examples from [IV] all fall within this class, on taking  $\phi(z) = e^{-z}$ ,  $\phi(z) = e^{-\pi z^2}$  and  $\phi(z) = \frac{1}{(z+1)^2}$  respectively.

## 2.1 A precise formulation of the validity condition

Letting the Taylor series for  $\phi$  around 0 be given by  $\phi(z) = \sum_{n=0}^{\infty} c_n z^n$  we have that

$$\phi(ju) = \sum_{n=0}^{\infty} c_n j^n u^n. \quad (4)$$

The Césaro array methodology places  $\phi(ju)$  at  $x = j$  and expands this Taylor series vertically in the y-direction to place the power  $c_n j^n u^n$  at  $(j, n)$  in the 2-d plane. Taking horizontal p-sums at each "height"  $n$  (i.e. degree  $u^n$ ) and writing  $X = k + \alpha$  as usual, we then invoke the *strong* Césaro asymptotic relationship that

$$\sum_{j=1}^k j^n \underset{\mathcal{C}}{\simeq} \frac{X^{n+1}}{n+1} + \zeta(-n) \quad . \quad (5)$$

Under our methodology we do not then Césaro-annihilate the terms  $c_n \frac{X^{n+1}}{n+1}$  at each height, but rather gather them separately to obtain an overall contribution from these "second-component pieces" of

$$\begin{aligned} \sum_{n=0}^{\infty} c_n \frac{X^{n+1}}{n+1} u^n &= \int_0^X \phi(uz) dz = \frac{1}{u} \int_0^{Xu} \phi(v) dv \\ &\xrightarrow{X \rightarrow \infty} \frac{1}{u} \int_0^{\infty} \phi(v) dv = \frac{L}{u}. \end{aligned} \quad (6)$$

Together with the collection of "first-component" terms,  $\zeta(-n) \cdot u^n$  for all heights  $n \in \mathbb{Z}_{\geq 0}$ , we obtain a power series expansion for  $f(u)$  near  $u = 0$  as

$$f(u) = \frac{L}{u} + \sum_{n=0}^{\infty} c_n \cdot \zeta(-n) \cdot u^n. \quad (7)$$

The claim of Césaro array methodology is that this is the correct asymptotic power series expansion for  $f(u)$  as  $u \rightarrow 0^+$  for any  $\phi$  in our specified class.

To make this an explicit mathematical statement, we recall from [III] that equation 5 has a more precise formulation. This is that

$$\sum_{j=1}^k j^n = \frac{X^{n+1}}{n+1} + \zeta(-n) - R_n(X) \quad (8)$$

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<sup>4</sup>Without loss of generality we consider only the case of  $u$  near 0 in this paper; everything may be easily adapted to the case  $u \rightarrow \infty$  or for  $u$  near any other point

where the remainder term,  $R_n(X)$ , has an exact expression in terms of the formal symbol  $\check{q}$  as

$$R_n(X) = \left(X - \check{q}\right)^n = \sum_{i=0}^n \binom{n}{i} X^{n-i} \check{q}^i = \sum_{i=0}^n \binom{n}{i} X^{n-i} \check{q}_i(X) \quad (9)$$

and  $R_n(X)$  is Césaro convergent to 0 under a pure power of the Césaro operator  $P$ , namely  $P^{n+1}$ ; i.e.  $R_n$  satisfies

$$P^{n+1} [R_n](X) \rightarrow 0 \quad \text{as } X \rightarrow \infty. \quad (10)$$

The heuristic argument we then made in [IV] was that, since the Césaro convergence of each  $R_n(X)$  involves only a pure power of  $P$ ; and since application of pure powers of  $P$  always tames function divergences and entails no concerns about accumulation points of non-unit eigenvalues of  $P$  in some more complicated regular function,  $q(u; P)$ ; so we should be able to *simultaneously* ignore the remainder terms,  $R_n(X)$ , at every height  $n$  and conclude that the power series expansion for  $f(u)$  given in equation 7 is valid, at least modulo  $\mathcal{S}_0(u)$ . It is this simultaneous omission of all these remainder terms,  $R_n(X)$ , modulo  $\mathcal{S}_0(u)$  which we need to justify mathematically in order to validate Césaro array methodology.

But based on equations 8 - 10 above, we see that for our specified class of cases we can instead gather together these remainder terms across all heights  $n$ , and hence re-write equation 7 as the following *exact* equation:

$$f(u) = \frac{L}{u} + \sum_{n=0}^{\infty} c_n \cdot \zeta(-n) \cdot u^n - \mathop{Clim}_{X \rightarrow \infty} R(X; u) \quad . \quad (11)$$

Here the overall remainder (or error) term,  $R(X; u)$ , is given formally by

$$R(X; u) := \sum_{n=0}^{\infty} R_n(X) \cdot u^n = \sum_{n=0}^{\infty} c_n \cdot \left(X - \check{q}\right)^n \cdot u^n = \phi((X - \check{q})u). \quad (12)$$

and is related to our original p-sum expansion by

$$R(X; u) := \sum_{j=1}^k \phi(ju) - \int_0^X \phi(uz) dz - \sum_{n=0}^{\infty} c_n \cdot \zeta(-n) \cdot u^n. \quad (13)$$

Thus our heuristic claim of the validity of Césaro array methodology becomes the following precise validity claim, at least for this class of cases:

**Claim 1 [Precise Validity Condition]:** Césaro array methodology is valid for  $\phi$  in our chosen class if and only if

$$\mathop{Clim}_{X \rightarrow \infty} R(X; u) = \mathop{Clim}_{X \rightarrow \infty} \phi((X - \check{q})u) \in \mathcal{S}_0(u) \quad \text{as } u \rightarrow 0^+. \quad (14)$$

## 2.2 Re-assessing our three key examples in terms of this validity condition

Before attempting to derive any *general* result from this reformulation, let us first re-consider each of the three successful examples of Césaro array methodology which we examined in [IV] in terms of this validity condition.

**Example 1** [ $f(u) := \sum_{j=1}^{\infty} e^{-ju}$ ]: This case arises in our class from taking  $\phi(z) = e^{-z}$ . We saw in [IV] that the power series which we obtain for  $f(u) = \sum_{j=1}^{\infty} e^{-ju}$  as  $u \rightarrow 0^+$  from applying our Césaro array methodology, namely

$$f(u) = \frac{1}{u} + \sum_{n=0}^{\infty} \frac{\zeta(-n)}{n!} u^n - \underset{X \rightarrow \infty}{\text{Clim}} R(X; u) \quad (15)$$

is in fact an exact relationship, with no residual remainder piece in  $\mathcal{S}_0(u)$  to consider; in other words that  $\underset{X \rightarrow \infty}{\text{Clim}} R(X; u) \equiv 0$  as a function of  $u$ , so that we have

$$f(u) = \frac{1}{u} + \sum_{n=0}^{\infty} \frac{\zeta(-n)}{n!} u^n \quad (16)$$

exactly as  $u \rightarrow 0^+$ . In [IV] we confirmed this by noting that for  $\text{Re}(u) > 0$ ,  $\sum_{j=1}^{\infty} e^{-ju}$  is in fact a convergent geometric progression in  $e^{-u}$  and so gives exactly  $\frac{1}{e^u - 1}$ . This is then easily seen to be given exactly by the convergent power series on the RHS in equation 16 on the complex plane with  $u = 0$  removed. The radius of convergence of this power series is  $2\pi$ , since  $\pm 2\pi i$  is the first non-trivial zero of the denominator in  $\frac{1}{e^u - 1}$ , and this is consistent with the fact that, by the functional equation for  $\zeta$ , we have  $\frac{\zeta(-2n-1)}{(2n+1)!} = 2^{-2n-1} \pi^{-2n-2} (-1)^{n-1} \zeta(2n+2)$  and thus

$$\lim_{n \rightarrow \infty} \left| \frac{\frac{\zeta(-2n-1)}{(2n+1)!}}{\frac{\zeta(-2n+1)}{(2n-1)!}} \right| = \lim_{n \rightarrow \infty} \left| \frac{2^{-2n-1} \pi^{-2n-2} \zeta(2n+2)}{2^{-2n+1} \pi^{-2n} \zeta(2n)} \right| = \frac{1}{4\pi^2}.$$

In this case, therefore, the Césaro array methodology is valid and, in terms of claim 1, the exactness of the resulting power series corresponds to the observation that, not only is  $\underset{X \rightarrow \infty}{\text{Clim}} R(X; u) = \underset{X \rightarrow \infty}{\text{Clim}} \phi((X - \overset{\vee}{q})u) = \underset{X \rightarrow \infty}{\text{Clim}} e^{-(X - \overset{\vee}{q})u}$  in  $\mathcal{S}_0(u)$ , it is actually (as noted above) identically zero for all  $u \in \mathbb{R}_{>0}$ . Can we see this directly, without recourse to the algebraic trick involving geometric progressions?

Well, by equation 13 we can see that  $R(X; u)$  converges *classically* as  $X \rightarrow \infty$  to its limiting value. Fixing  $u$  arbitrary and writing this limiting value as  $r(u)$ , it follows that for all  $\epsilon > 0$  there exists  $X_\epsilon$  sufficiently large that  $|r(u) - R(X; u)| < \epsilon$  whenever  $X > X_\epsilon$ .

But consider any such integer-valued  $X$ , i.e.  $X = k$  with  $\alpha = 0$  and  $k > X_\epsilon$ . Then since  $\overset{\vee}{q}_n(0) = (-1)^n \zeta(-n)$  and the expression for  $R(X; u) = e^{-(X - \overset{\vee}{q})u}$

splits algebraically we have

$$e^{-(X-\check{q})u} \Big|_{X=k} = \left( e^{-Xu} \cdot e^{\check{q}u} \right) \Big|_{X=k} = e^{-ku} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n \zeta(-n)}{n!} u^n \quad (17)$$

and it follows that  $R(k; u)$  separates into the product

$$R(k; u) = e^{-ku} \tilde{f}(u) \quad (18)$$

where  $\tilde{f}(u) = \sum_{n=0}^{\infty} \frac{(-1)^n \zeta(-n)}{n!} u^n$  is actually the function  $\frac{1}{e^u - 1} - \frac{1}{u}$  which we have analysed above and which is smooth and bounded on the disc  $|u| < 2\pi$ .

It follows that for any fixed  $u$  with  $|u| < 2\pi$  we have

$$\lim_{X \rightarrow \infty} R(X; u) = \lim_{k \rightarrow \infty} R(k; u) = \tilde{f}(u) \cdot \lim_{k \rightarrow \infty} e^{-ku} = 0 \quad . \quad (19)$$

We thus conclude immediately that  $r(u)$  must be identically zero for such  $u$ , i.e.

$$\mathcal{C}lim_{X \rightarrow \infty} R(X; u) = 0 \quad \text{for all } u \text{ such that } |u| < 2\pi \quad (20)$$

as claimed. This directly validates the condition in Claim 1 and hence confirms the validity of our Césaro array methodology in this example, without invoking the geometric-progression trick used previously - and in so doing explains why the Césaro array in this case gave an *exact* power series for the sum-function.

**Example 2** [ $\tilde{G}(u) := \frac{1}{2} + \sum_{j=1}^{\infty} e^{-\pi j^2 u^2}$ ]: This example arises in our class from taking  $\phi(z) = e^{-\pi z^2}$  and using Césaro array methodology we derived in [IV] the simple power series expansion for it that

$$\tilde{G}(u) = \frac{1}{2} \frac{1}{u} + \mathcal{S}_0(u) \quad \text{as } u \rightarrow 0^+ \quad . \quad (21)$$

In [IV] we were able to verify the correctness of this expansion by quoting the functional equation for  $\tilde{G}$ , namely that  $\tilde{G}(u) = \frac{1}{u} \tilde{G}\left(\frac{1}{u}\right)$ , from which equation 21 follows immediately; as does the fact that, unlike in example 1, the  $\mathcal{S}_0(u)$  remainder piece is *not* identically zero in this case.

Can we establish the validity of our Césaro array methodology in this case, and hence the validity of equation 21, without using this functional equation and instead by validating condition 14 in Claim 1? And, if so, can we demonstrate the fact that the remainder piece  $r(u) := \mathcal{C}lim_{X \rightarrow \infty} R(X; u)$  is not identically zero, but rather only in  $\mathcal{S}_0(u)$ , as part of this demonstration?

Well, in this instance we still have the same *classical* convergence of  $R(X; u) = e^{-\pi(X-\check{q})^2 u^2}$  to its limiting value  $r(u)$ . However, we cannot algebraically split  $e^{-\pi(X-\check{q})^2 u^2}$  as we did in example 1,<sup>5</sup> and so we need to understand this quantity and its Césaro limit as  $X \rightarrow \infty$  by different means. The key is to recall

<sup>5</sup>Which is just as well since we know that  $r(u)$  is not identically zero!

from [III] that we have the following derivative relationship for  $\frac{d}{dX}(X - \check{q})^n$ :

$$\frac{d}{dX}(X - \check{q})^n = X^n - X^n \cdot \sum_{j=1}^{\infty} \delta_j(X). \quad (22)$$

It follows that

$$\frac{d}{dX} e^{-\pi(X - \check{q})^2 u^2} = e^{-\pi X^2 u^2} - e^{-\pi X^2 u^2} \cdot \sum_{j=1}^{\infty} \delta_j(X). \quad (23)$$

and so, on integrating from 0 to  $X$ , we have that

$$e^{-\pi(X - \check{q})^2 u^2} - e^{-\pi(X - \check{q})^2 u^2} \Big|_{X=0} = \int_0^X e^{-\pi \tilde{X}^2 u^2} d\tilde{X} - \sum_{j=1}^k e^{-\pi j^2 u^2}. \quad (24)$$

Since  $\check{q}_{2n}(0) = \zeta(-2n) = 0$  for all  $n \in \mathbb{Z}_{\geq 1}$ , the LHS here is just  $e^{-\pi(X - \check{q})^2 u^2} + \frac{1}{2}$ . As for the RHS, to understand it we will use the Euler-McLaurin sum formula, but now the following version (from [6]), which has an explicit form for the error term, one which allows us to comprehend in detail its asymptotic behaviour as  $X \rightarrow \infty$ :

**Euler-McLaurin Theorem:** *Let  $x_0$  be a real number and  $f : [x_0, \infty) \rightarrow \mathbb{C}$  a continuously differentiable function. Then we have for all integers  $n \geq m \geq x_0$  that*

$$\sum_{j=m}^n f(j) = \frac{1}{2} (f(m) + f(n)) + \int_m^n f(x) dx + \int_m^n \check{q}_0(x) f'(x) dx. \quad (25)$$

Applying this with  $m = 0$ ,  $n = k$  and  $f(\tilde{X}) = e^{-\pi \tilde{X}^2 u^2}$  it follows, after rearranging, that we have

$$\int_0^k e^{-\pi \tilde{X}^2 u^2} d\tilde{X} - \sum_{j=1}^k e^{-\pi j^2 u^2} = \frac{1}{2} - \frac{1}{2} e^{-\pi k^2 u^2} + 2\pi u^2 \int_0^k \check{q}_0(\tilde{X}) \tilde{X} e^{-\pi \tilde{X}^2 u^2} d\tilde{X}. \quad (26)$$

Now, note that both  $\int_k^X e^{-\pi \tilde{X}^2 u^2} d\tilde{X} \rightarrow 0$  and  $\frac{1}{2} e^{-\pi k^2 u^2} \rightarrow 0$  as  $X \rightarrow \infty$  for any  $u \in \mathbb{R}_{>0}$ . Also, making the substitution  $v = u\tilde{X}$  we have that

$$2\pi u^2 \int_0^k \check{q}_0(\tilde{X}) \tilde{X} e^{-\pi \tilde{X}^2 u^2} d\tilde{X} = 2\pi \int_0^{uk} \check{q}_0\left(\frac{v}{u}\right) v e^{-\pi v^2} dv$$

and the oscillatory, saw-tooth nature of  $\check{q}_0\left(\frac{v}{u}\right)$  means this is a rapidly convergent integral. Thus we may let  $X \rightarrow \infty$  and deduce in equation 24 that

$$\lim_{X \rightarrow \infty} \left\{ \int_0^X e^{-\pi \tilde{X}^2 u^2} d\tilde{X} - \sum_{j=1}^k e^{-\pi j^2 u^2} \right\} = \frac{1}{2} + 2\pi \int_0^{\infty} \check{q}_0\left(\frac{v}{u}\right) v e^{-\pi v^2} dv. \quad (27)$$

To evaluate the integral on the RHS we then recall the Fourier series decomposition of  $\check{q}_0(x)$  from [III] (or alternatively from [6] again), namely that

$$\check{q}_0(x) = -2 \sum_{n=1}^{\infty} \frac{\sin(2\pi n x)}{2\pi n} = \frac{-1}{2\pi i} \sum_{n=1}^{\infty} \frac{1}{n} (e^{2\pi i n x} - e^{-2\pi i n x}) \quad (x \in \mathbb{R} \setminus \mathbb{Z}). \quad (28)$$

Applying the dominated convergence theorem to reverse the order of summation and integration, it follows in equation 27 that we have

$$\begin{aligned} & \lim_{X \rightarrow \infty} \left\{ \int_0^X e^{-\pi \tilde{X}^2 u^2} d\tilde{X} - \sum_{j=1}^k e^{-\pi j^2 u^2} \right\} \\ &= \frac{1}{2} + i \sum_{n=1}^{\infty} \frac{1}{n} \left\{ \int_0^{\infty} v e^{-\pi v^2} e^{2\pi i n \frac{v}{u}} dv - \int_0^{\infty} v e^{-\pi v^2} e^{-2\pi i n \frac{v}{u}} dv \right\} \\ &= \frac{1}{2} - i \sum_{n=1}^{\infty} \frac{1}{n} \int_{-\infty}^{\infty} v e^{-\pi v^2} e^{-2\pi i n \frac{v}{u}} dv \end{aligned} \quad (29)$$

on using the substitution  $w = -v$  in the first of the two integrals on the RHS and then combining it with the second integral.

Now  $\int_{-\infty}^{\infty} e^{-\pi v^2} e^{-2\pi i n \frac{v}{u}} dv = \mathcal{F} \left[ e^{-\pi v^2} \right] \left( \frac{n}{u} \right)$  where  $\mathcal{F}$  is the Fourier transform operator defined by

$$\mathcal{F}[f](\xi) := \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} dx \quad (30)$$

and it is well-known (or else see [7]) that  $e^{-\pi v^2}$  is a fixed-point of this operator, i.e. that

$$\mathcal{F} \left[ e^{-\pi v^2} \right] (\xi) = \int_{-\infty}^{\infty} e^{-\pi v^2} e^{-2\pi i \xi v} dv = e^{-\pi \xi^2}. \quad (31)$$

It follows, on noting that  $v e^{-\pi v^2} e^{-2\pi i \xi v} = \frac{i}{2\pi} \frac{d}{d\xi} \left( e^{-\pi v^2} e^{-2\pi i \xi v} \right)$  and that we may differentiate under the integral, that

$$\frac{1}{n} \int_{-\infty}^{\infty} v e^{-\pi v^2} e^{-2\pi i n \frac{v}{u}} dv = \frac{i}{2\pi n} \frac{d}{d\xi} \left( e^{-\pi \xi^2} \right) \Big|_{\xi = \frac{n}{u}} = -\frac{i}{u} e^{-\pi \frac{n^2}{u^2}} \quad (32)$$

and hence, in equation 29, we deduce that

$$\lim_{X \rightarrow \infty} \left\{ \int_0^X e^{-\pi \tilde{X}^2 u^2} d\tilde{X} - \sum_{j=1}^k e^{-\pi j^2 u^2} \right\} = \frac{1}{2} - \frac{1}{u} \sum_{j=1}^{\infty} e^{-\pi \frac{j^2}{u^2}} = \frac{1}{2} + \mathcal{S}_0(u) \quad (33)$$

It follows in equation 24 that we do indeed have that

$$\mathcal{C}limR(X; u) = \lim_{X \rightarrow \infty} e^{-\pi (X - \check{q})^2 u^2} = -\frac{1}{u} \sum_{j=1}^{\infty} e^{-\pi \frac{j^2}{u^2}} = \mathcal{S}_0(u) \quad . \quad (34)$$

We have thus established the validity of the Césaro array methodology we applied in this case in [IV] by verifying the validity condition in Claim 1 directly. In doing so, we have verified explicitly that the remainder piece, while in  $\mathcal{S}_0(u)$ , is not identically zero as it was in our first example.

**Comment:** In the above derivation we have not only shown that  $\mathop{Clim}_{X \rightarrow \infty} R(X; u) = \mathcal{S}_0(u)$ , we have in fact derived its precise form as  $-\frac{1}{u} \sum_{j=1}^{\infty} e^{-\pi \frac{j^2}{u^2}}$ . Our Césaro array calculation in full (per equation 11) thus gives us exactly that

$$\tilde{G}(u) = \frac{1}{2} \frac{1}{u} + \frac{1}{u} \sum_{j=1}^{\infty} e^{-\pi \frac{j^2}{u^2}} = \frac{1}{u} \tilde{G}\left(\frac{1}{u}\right) \quad (35)$$

which is, of course, the functional equation for  $\tilde{G}$ . Thus, in this case, Césaro array methodology not only gives us very easily the unexpectedly simple power series asymptotics for  $\tilde{G}(u)$  as  $u \rightarrow 0^+$ . It also gives us the full functional equation for  $\tilde{G}$  when we use a combination of the precise form of the Euler-McLaurin sum formula and Fourier theory for  $\check{q}_0$  to calculate  $\mathop{Clim}_{X \rightarrow \infty} R(X; u)$  (as part of confirming the validity condition for the methodology given in Claim 1).

**Example 3** [ $f(u) := -\frac{1}{u} + \sum_{j=1}^{\infty} \frac{1}{(ju+1)^2}$ ]: This example arises in our class from taking  $\phi(z) = \frac{1}{(z+1)^2}$ . Using Césaro array methodology we were able to show in [IV] that it has the asymptotic expansion

$$f(u) = \sum_{j=0}^{\infty} B_{j+1} u^j + \mathcal{S}_0(u) \quad \text{as } u \rightarrow 0^+ \quad (36)$$

where  $B_{j+1} = (-1)^j (j+1) \zeta(-j)$  are the Bernoulli numbers. This is a true asymptotic expansion in the sense that, although  $f$  is well-defined for all  $u \in \mathbb{R}_{>0}$ , this power series has radius of convergence 0 and is only well-defined as an asymptotic series as  $u \rightarrow 0^+$ .

In [IV] we did not supply any independent validation of this new example by alternative means (e.g. using geometric progressions or functional equations) and so it remains to confirm its validity by directly checking the condition given in Claim 1. To do this we follow the same approach used in example 2 as far as we can, before needing to take a different calculational path at the end.

Specifically, in this case we have

$$R(X; u) = \phi\left((X - \check{q})u\right) = \frac{1}{\left((X - \check{q})u + 1\right)^2} \quad (37)$$

and this satisfies

$$\frac{d}{dX} R(X; u) = \frac{1}{(Xu + 1)^2} - \frac{1}{(Xu + 1)^2} \cdot \sum_{j=1}^{\infty} \delta_j(X) \quad . \quad (38)$$

On integrating from 0 to  $X$  we get

$$\left. \frac{1}{((X - \check{q})u + 1)^2} - \frac{1}{((X - \check{q})u + 1)^2} \right|_{X=0} = \int_0^X \frac{1}{(\tilde{X}u + 1)^2} d\tilde{X} - \sum_{j=1}^k \frac{1}{(ju + 1)^2}. \quad (39)$$

Now the Taylor series for  $\phi(z) = \frac{1}{(z+1)^2}$  around 0 is  $\phi(z) = 1 - 2z + 3z^2 - 4z^3 + \dots$  and so, since  $\check{q}_n(0) = \zeta(-n)$  for all  $n \in \mathbb{Z}_{\geq 0}$  it follows on the LHS in equation 39 that

$$\left. \frac{1}{((X - \check{q})u + 1)^2} \right|_{X=0} = \sum_{n=0}^{\infty} (-1)^n (n+1) \zeta(-n) u^n = -\frac{1}{2} + \frac{1}{6}u - \frac{1}{30}u^3 + \dots \quad (40)$$

And by applying the Euler-McLaurin sum formula as we did in the previous example, we obtain on the RHS in equation 39 that

$$\begin{aligned} \text{Clim}_{X \rightarrow \infty} \left\{ \begin{array}{l} \int_0^X \frac{1}{(\tilde{X}u + 1)^2} d\tilde{X} \\ - \sum_{j=1}^k \frac{1}{(ju + 1)^2} \end{array} \right\} &= \frac{1}{2} + 2u \lim_{k \rightarrow \infty} \int_0^k \check{q}_0(\tilde{X}) \cdot \frac{1}{(\tilde{X}u + 1)^3} d\tilde{X} \\ &= \frac{1}{2} + 2 \lim_{k \rightarrow \infty} \int_0^{ku} \check{q}_0\left(\frac{v}{u}\right) \cdot \frac{1}{(v + 1)^3} dv \\ &= \frac{1}{2} - 4 \int_0^{\infty} \sum_{n=1}^{\infty} \left( \frac{1}{2\pi n} \right) \cdot \frac{\sin\left(\frac{2\pi n v}{u}\right)}{(v + 1)^3} dv \\ &= \frac{1}{2} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} \frac{\sin\left(\frac{2\pi n v}{u}\right)}{(v + 1)^3} dv. \quad (41) \end{aligned}$$

Since  $\frac{1}{(v+1)^3}$  is not an even function, however, we cannot convert this integral into a Fourier transform by writing  $\sin\left(\frac{2\pi n v}{u}\right) = \frac{1}{2i} (e^{2\pi i n \frac{v}{u}} - e^{-2\pi i n \frac{v}{u}})$ . Instead we need to draw on the following integral identity involving the sine-integral and cos-integral functions,  $si(x)$  and  $ci(x)$  (see [7], pp423):

$$\int_0^{\infty} \frac{\sin(av)}{v + \beta} dv = ci(a\beta) \cdot \sin(a\beta) - \cos(a\beta) \cdot si(a\beta) \quad \text{for } a > 0 \text{ and } |\arg(\beta)| < \pi \quad (42)$$

where

$$si(x) := - \int_x^{\infty} \frac{\sin t}{t} dt = \int_0^x \frac{\sin t}{t} dt - \frac{\pi}{2} \quad (43)$$

and

$$ci(x) := - \int_x^{\infty} \frac{\cos t}{t} dt = \gamma + \ln x + \int_0^x \frac{(\cos t - 1)}{t} dt. \quad (44)$$

Differentiating twice under the integral w.r.t  $\beta$ , this yields that

$$2 \int_0^{\infty} \frac{\sin(av)}{(v + \beta)^3} dv = \frac{a}{\beta} - a^2 \cdot \{ci(a\beta) \cdot \sin(a\beta) - \cos(a\beta) \cdot si(a\beta)\} \quad (45)$$

and taking  $a = \frac{2\pi n}{u}$  and  $\beta = 1$  we thus get that

$$\int_0^\infty \frac{\sin\left(\frac{2\pi n v}{u}\right)}{(v+1)^3} dv = \frac{1}{2} \left\{ \frac{2\pi n}{u} - \frac{4\pi^2 n^2}{u^2} \cdot \left\{ \begin{array}{l} ci\left(\frac{2\pi n}{u}\right) \cdot \sin\left(\frac{2\pi n}{u}\right) \\ - \cos\left(\frac{2\pi n}{u}\right) \cdot si\left(\frac{2\pi n}{u}\right) \end{array} \right\} \right\} . \quad (46)$$

Now for  $x$  large, we have the following well-known asymptotic expansions for  $si(x)$  and  $ci(x)$ :

$$si(x) = -\frac{\cos x}{x} \left\{ 1 - \frac{2!}{x^2} + \frac{4!}{x^4} - \dots \right\} - \frac{\sin x}{x} \left\{ \frac{1}{x} - \frac{3!}{x^3} + \frac{5!}{x^5} - \dots \right\} \quad (47)$$

and

$$ci(x) = \frac{\sin x}{x} \left\{ 1 - \frac{2!}{x^2} + \frac{4!}{x^4} - \dots \right\} - \frac{\cos x}{x} \left\{ \frac{1}{x} - \frac{3!}{x^3} + \frac{5!}{x^5} - \dots \right\} . \quad (48)$$

As  $u \rightarrow 0^+$ ,  $\frac{2\pi n}{u} \rightarrow \infty$  and so we can apply these asymptotic expansions to deduce, after some elementary algebra, that

$$\left\{ \begin{array}{l} ci\left(\frac{2\pi n}{u}\right) \cdot \sin\left(\frac{2\pi n}{u}\right) \\ - \cos\left(\frac{2\pi n}{u}\right) \cdot si\left(\frac{2\pi n}{u}\right) \end{array} \right\} = \frac{u}{2\pi n} \left\{ 1 - \frac{2!u^2}{(2\pi n)^2} + \frac{4!u^4}{(2\pi n)^4} - \dots \right\} + \mathcal{S}_0(u) \quad (49)$$

and it then follows in equation 46 that

$$\int_0^\infty \frac{\sin\left(\frac{2\pi n v}{u}\right)}{(v+1)^3} dv = \frac{1}{2} \left( \frac{2\pi n}{u} \right) \left\{ \frac{2!u^2}{(2\pi n)^2} - \frac{4!u^4}{(2\pi n)^4} + \dots \right\} + \mathcal{S}_0(u) . \quad (50)$$

Hence in equation 41 we obtain that

$$\begin{aligned} \mathop{Clim}_{X \rightarrow \infty} \left\{ \begin{array}{l} \int_0^X \frac{1}{(\tilde{X}u+1)^2} d\tilde{X} \\ - \sum_{j=1}^k \frac{1}{(ju+1)^2} \end{array} \right\} &= \frac{1}{2} - \frac{2}{u} \sum_{n=1}^\infty \left\{ \frac{2!u^2}{(2\pi)^2 n^2} - \frac{4!u^4}{(2\pi)^4 n^4} + \dots \right\} + \mathcal{S}_0(u) \\ &= \frac{1}{2} - \frac{2}{u} \left\{ \frac{2!\zeta(2)}{2^2 \pi^2} u^2 - \frac{4!\zeta(4)}{2^4 \pi^4} u^4 + \dots \right\} + \mathcal{S}_0(u) \\ &= \frac{1}{2} + \sum_{j=1}^\infty (j+1)\zeta(-j)u^j + \mathcal{S}_0(u) \end{aligned} \quad (51)$$

where, in going from the second last to the last line, we have used the functional equation for  $\zeta$  in the same fashion as we did in example 1.

Finally then, in equation 39 it follows that

$$\begin{aligned} \mathop{Clim}_{X \rightarrow \infty} R(X; u) &= \mathop{Clim}_{X \rightarrow \infty} \frac{1}{((X - \check{q})u + 1)^2} \\ &= \sum_{j=0}^\infty (-1)^j (j+1)\zeta(-j)u^j + \frac{1}{2} + \sum_{j=1}^\infty (j+1)\zeta(-j)u^j + \mathcal{S}_0(u) \\ &= \mathcal{S}_0(u) . \end{aligned} \quad (52)$$

This proves that the validity condition in Claim 1 is satisfied, and thereby validates the correctness of the overall asymptotic expansion we obtained in [IV] (equation 36 here) for  $f(u)$  using our Césaro array methodology.

Overall we have confirmed the validity of the power series expansions we obtained in [IV] for all three of our examples, by confirming the validity condition derived in Claim 1 in each case. The Césaro array methodology provides a very quick path to these expansions, and in all three cases our validation of them has been done using machétes which we have fashioned from our Césaro derivative identity, from the precise form of the Euler-McLaurin sum formula, and from the Fourier series decomposition of  $\check{q}_0(\alpha)$  to clear a path leading to a direct confirmation of the validity condition in claim 1 (equation 14). This suggests these same tools may in fact be sufficient to confirm the validity condition in *general* for the entire class of cases we have considered, although the manner in which the working was obliged to diverge between examples 2 and 3 suggests the clearance work may not be straightforward. We now turn to trying to derive a first such general result along these lines.

### 2.3 A first general result on the validity of Césaro array methodology

For simplicity - in order to make the derivation of a first general result as free from complications as possible - we start by restricting our class of cases a little further to make  $\phi$  as nice as we could reasonably ask. Specifically, we consider the sub-class where  $\phi$  is smooth on  $[0, \infty)$  (meaning  $\phi$  is smooth on  $[x_0, \infty)$  for some  $x_0 \in \mathbb{R}_{<0}$ ) and is Schwartzian near  $\infty$ , i.e.  $\phi(z) \in \mathcal{S}_\infty(z)$  so that  $\phi$  and all its derivatives decay faster than any power of  $z$  as  $z \rightarrow \infty$ .

Clearly such a  $\phi$  satisfies the existing conditions 1 - 3 for the class we have been examining, and so this superseding condition delineates a sub-class of the one we have been considering so far. This sub-class covers examples 1 and 2 from section 2.2 but not example 3 and, once we have our result, we will turn in section 2.4 to an initial discussion of where this tighter condition could be relaxed and thus where our general validity result might be extended.

Since  $\phi$  is smooth in a neighbourhood of 0, let its Taylor series around 0 be given by  $\phi(z) = \sum_{n=0}^{\infty} c_n z^n$  as before. Then our Césaro array methodology gives us that  $f(u) := \sum_{j=1}^{\infty} \phi(ju)$  satisfies the asymptotic relations 11 - 13 and we wish to show, per Claim 1, that  $\mathop{Clim}_{X \rightarrow \infty} R(X; u) = \mathcal{S}_0(u)$  as  $u \rightarrow 0^+$ .

Resharpener our machétes, we proceed along the track we began clearing in examples 2 and 3 as far as we can.

Since  $R(X; u) = \phi((X - \check{q})u) = \sum_{n=0}^{\infty} c_n \cdot (X - \check{q})^n \cdot u^n$  it follows by our derivative identity equation 22 that we have

$$\frac{d}{dX} R(X; u) = \phi(Xu) - \phi(Xu) \cdot \sum_{j=1}^{\infty} \delta_j(X) \quad (53)$$

and hence, integrating from 0 to  $X$ , we have that

$$R(X; u) - R(X; u) \Big|_{X=0} = \int_0^X \phi(\tilde{X}u) d\tilde{X} - \sum_{j=1}^k \phi(ju). \quad (54)$$

Now since  $\check{q}_n(0) = (-1)^n \zeta(-n)$  it follows that

$$R(X; u) \Big|_{X=0} = \sum_{n=0}^{\infty} c_n \cdot (X - \check{q})^n \Big|_{X=0} \cdot u^n = \sum_{n=0}^{\infty} c_n \zeta(-n) u^n \quad (55)$$

and so

$$R(X; u) = \sum_{n=0}^{\infty} c_n \zeta(-n) u^n + \left\{ \int_0^X \phi(\tilde{X}u) d\tilde{X} - \sum_{j=1}^k \phi(ju) \right\}. \quad (56)$$

But, by the Euler-McLaurin sum formula equation 25, applied as we did for examples 2 and 3 in section 2.2, we have that

$$\int_0^k \phi(\tilde{X}u) d\tilde{X} - \sum_{j=1}^k \phi(ju) = \frac{1}{2} \phi(0) - \frac{1}{2} \phi(ku) - u \int_0^k \check{q}_0(\tilde{X}) \phi'(\tilde{X}u) d\tilde{X}. \quad (57)$$

Thus, since  $\frac{1}{2} \phi(0) = \frac{1}{2} c_0$  and  $\lim_{k \rightarrow \infty} \phi(ku) = 0$ ; and since  $\lim_{X \rightarrow \infty} \int_k^X \phi(\tilde{X}u) d\tilde{X} = 0 = \lim_{X \rightarrow \infty} \int_k^X \check{q}_0(\tilde{X}) \phi'(\tilde{X}u) d\tilde{X}$  it follows that we have

$$\mathit{Clim}_{X \rightarrow \infty} \left\{ \int_0^X \phi(\tilde{X}u) d\tilde{X} - \sum_{j=1}^k \phi(ju) \right\} = \frac{1}{2} c_0 - u \int_0^{\infty} \check{q}_0(\tilde{X}) \phi'(\tilde{X}u) d\tilde{X} \quad (58)$$

where we have passed from  $\mathit{Clim}_{X \rightarrow \infty}$  to  $\lim_{X \rightarrow \infty}$  in the integral on the RHS since it is clearly *classically* convergent to its ( $u$ -dependent) limiting value.

Making the familiar substitution  $v = \tilde{X}u$  and then invoking the Fourier series decomposition of  $\check{q}_0(\frac{v}{u})$  as we have before, this latter integral then becomes

$$\begin{aligned} u \int_0^{\infty} \check{q}_0(\tilde{X}) \phi'(\tilde{X}u) d\tilde{X} &= -2 \int_0^{\infty} \sum_{n=1}^{\infty} \frac{\sin(2\pi n \frac{v}{u})}{2\pi n} \phi'(v) dv \quad (59) \\ &= \int_0^{\infty} \sum_{n=1}^{\infty} \frac{i}{2\pi n} \{e^{2\pi i n \frac{v}{u}} - e^{-2\pi i n \frac{v}{u}}\} \phi'(v) dv. \quad (60) \end{aligned}$$

To proceed, we now consider the sub-cases of  $\phi$  even and  $\phi$  odd separately.

**Sub-Case 1 [ $\phi$  even]:** In this case  $c_n = 0$  for all  $n$  odd and since  $\zeta(-n) = 0$  for all  $n \in \mathbb{Z}_{\geq 1}$  even, it follows on combining equations 56 and 58 that we have

$$\mathit{Clim}_{X \rightarrow \infty} R(X; u) = -\frac{i}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} \{e^{2\pi i n \frac{v}{u}} - e^{-2\pi i n \frac{v}{u}}\} \phi'(v) dv \quad (61)$$

where the reversal of integration and summation is clearly justified since  $\phi$  is Schwartzian. But letting  $w = -v$  it follows from the oddness of  $\phi'(v)$  that

$$\int_0^\infty e^{2\pi i n \frac{v}{u}} \phi'(v) dv = - \int_{-\infty}^0 e^{-2\pi i n \frac{w}{u}} \phi'(w) dw \quad (62)$$

and so in equation 61 we get that

$$\mathop{Clim}_{X \rightarrow \infty} R(X; u) = \frac{i}{2\pi} \sum_{n=1}^\infty \frac{1}{n} \mathcal{F}[\phi']\left(\frac{n}{u}\right). \quad (63)$$

Since the Fourier transform of a Schwartzian function is Schwartzian<sup>6</sup> and a countable sum of Schwartzian functions is Schwartzian, and since  $\frac{n}{u} \rightarrow \infty$  as  $u \rightarrow 0^+$ , it follows that

$$\mathop{Clim}_{X \rightarrow \infty} R(X; u) = \mathcal{S}_0(u) \quad \text{as } u \rightarrow 0^+ \quad . \quad (64)$$

We have thus verified that the validity condition of Claim 1 is indeed satisfied for this sub-case where  $\phi$  is even and Schwartzian.

**Sub-Case 2 [ $\phi$  odd]:** In this case, equations 56 and 58 combine merely to say that

$$\mathop{Clim}_{X \rightarrow \infty} R(X; u) = \sum_{m \text{ odd}} c_m \zeta(-m) u^m - \frac{i}{2\pi} \sum_{n=1}^\infty \frac{1}{n} \int_0^\infty \left\{ \begin{array}{c} e^{2\pi i n \frac{v}{u}} \\ -e^{-2\pi i n \frac{v}{u}} \end{array} \right\} \phi'(v) dv. \quad (65)$$

We cannot rearrange this to generate an integral representing a Fourier transform, and the calculation which follows for this case is not a rigorous derivation (as it was for sub-case 1), but it suffices to indicate why we still expect to have  $\mathop{Clim}_{X \rightarrow \infty} R(X; u) = \mathcal{S}_0(u)$ . We are confident that the machinery of distributional analysis could be deployed to take the preliminary path clearance which follows, and transform it into a properly surveyed and navigable mathematical track.

Continuing with our machétes then, we proceed by noting that as  $u \rightarrow 0^+$ ,  $e^{\pm 2\pi i n \frac{v}{u}}$  oscillates more and more rapidly, so that the integrals in equation 65 on any sub-domain  $[x_0, \infty)$  excising 0 (i.e.  $x_0 > 0$ ) will give a contribution which is in  $\mathcal{S}_0(u)$ .

Thus, modulo  $\mathcal{S}_0(u)$ , we may focus on a neighbourhood of 0 where  $\phi'(v)$  is given by its Taylor series  $\phi'(v) = c_1 + 3c_3 v^2 + 5c_5 v^4 + \dots$ . Now

$$\begin{aligned} \int_0^\infty e^{-2\pi i \xi v} v^k dv &= \left(\frac{i}{2\pi}\right)^k \left(\frac{d}{d\xi}\right)^k \int_0^\infty e^{-2\pi i \xi v} dv \\ &= \left(\frac{i}{2\pi}\right)^k \left(\frac{d}{d\xi}\right)^k \mathcal{F}[\text{sgn}(v)](\xi) \\ &= \left(\frac{i}{2\pi}\right)^k \left(\frac{d}{d\xi}\right)^k \left[ \frac{1}{2\pi i \xi} \right] = \frac{-i^{k+1}}{(2\pi)^{k+1}} (-1)^k \frac{k!}{\xi^{k+1}} \quad (66) \end{aligned}$$

<sup>6</sup>This is the starting point for the development of the theory of tempered distributions

on using the well-known Fourier transform of the one-sided Heaviside function  $\text{sgn}(v)$ . It follows that

$$\int_0^\infty e^{-2\pi i n \frac{v}{u}} v^k dv = \frac{(-1)^{k+1} i^{k+1}}{(2\pi)^{k+1}} \frac{k!}{n^{k+1}} u^{k+1} \quad (67)$$

and

$$\int_0^\infty e^{2\pi i n \frac{v}{u}} v^k dv = \frac{i^{k+1}}{(2\pi)^{k+1}} \frac{k!}{n^{k+1}} u^{k+1}. \quad (68)$$

Thus for  $m = 2l + 1$  odd we have that

$$\begin{aligned} & -\frac{i}{2\pi} (2l+1) c_{2l+1} \sum_{n=1}^\infty \frac{1}{n} \left\{ \int_0^\infty e^{2\pi i n \frac{v}{u}} v^{2l} dv - \int_0^\infty e^{-2\pi i n \frac{v}{u}} v^{2l} dv \right\} \\ = & -\frac{i}{2\pi} (2l+1) c_{2l+1} \sum_{n=1}^\infty \left\{ \frac{(-1)^l i}{(2\pi)^{2l+1}} \frac{(2l)!}{n^{2l+2}} + \frac{(-1)^l i}{(2\pi)^{2l+1}} \frac{(2l)!}{n^{2l+2}} \right\} u^{2l+1} \\ = & \frac{(-1)^l}{2^{2l+1} \pi^{2l+2}} \cdot c_{2l+1} \cdot (2l+1)! \cdot \zeta(2l+2) \cdot u^{2l+1} \\ = & -c_{2l+1} \zeta(-2l-1) u^{2l+1} \end{aligned} \quad (69)$$

on using the functional equation for  $\zeta$  in the usual way in the last step; and so

$$\begin{aligned} & -\frac{i}{2\pi} \sum_{n=1}^\infty \frac{1}{n} \left\{ \int_0^\infty \{c_1 + 3c_3 v^2 + 5c_5 v^4 + \dots\} \{e^{2\pi i n \frac{v}{u}} - e^{-2\pi i n \frac{v}{u}}\} dv \right\} \\ = & -\sum_{m \text{ odd}} c_m \zeta(-m) u^m. \end{aligned} \quad (70)$$

Now this calculation has involved taking the Taylor series for  $\phi'(v)$  applicable in a neighbourhood of 0 and integrating it term by term, but on all of  $[0, \infty)$ , not just this neighbourhood. However, as  $u \rightarrow 0^+$ ,  $\xi = \frac{n}{u} \rightarrow \infty$  and so the error in this computation arising from this mismatch becomes small very rapidly, indeed is itself in  $\mathcal{S}_0(u)$ <sup>7</sup>. As such it follows in equation 65 that we have

$$\begin{aligned} \lim_{X \rightarrow \infty} R(X; u) &= \sum_{m \text{ odd}} c_m \zeta(-m) u^m - \sum_{m \text{ odd}} c_m \zeta(-m) u^m + \mathcal{S}_0(u) \\ &= \mathcal{S}_0(u) \quad \text{as } u \rightarrow 0^+ \end{aligned} \quad (71)$$

and thus we have proved (or rather sketch-proved) that the validity condition of Claim 1 is also satisfied for this sub-case of  $\phi$  Schwartzian and odd.

Overall, then, we have the following first general result regarding the validity of Césaro array methodology:

**Result 1:** *If  $\phi$  is Schwartzian on  $[0, \infty)$  (meaning  $\phi$  is smooth on  $(x_0, \infty)$ )*

<sup>7</sup>this claim is one area where we acknowledge that greater rigour is clearly ultimately required!

for some  $x_0 \in \mathbb{R}_{<0}$  and lies in  $\mathcal{S}_\infty$ ) then the validity condition in Claim 1 is satisfied and so Césaro array methodology may be successfully applied to derive the asymptotic expansion for  $\sum_{j=1}^{\infty} \phi(ju)$  given in equation 7.

This has been rigorously proven for  $\phi$  even and a sketch-proof has been supplied for  $\phi$  odd. Once this latter is also rigourised, the proof for general Schwartzian  $\phi$  would be complete, since any general  $\phi$  can be split as a sum of even and odd pieces.

## 2.4 Extending the general domain of applicability of Césaro array methodology

The condition on  $\phi$  in Result 1 (that it be Schwartzian on  $[0, \infty)$ ) covers rigorously the case of example 2 in section 2.2; and also covers the case of example 1 if we allow ourselves to be a little more flexible in our standards and accept the sketch argument justifying the result for  $\phi$  odd. But it does not cover the case of example 3, where  $\phi(z) = \frac{1}{(z+1)^2}$  is not Schwartzian. It is also a highly restrictive condition in principle; restrictive even within the realm of classical convergence, let alone from the perspective of the Césaro convergence worldview.

It thus seems certain that Result 1 can be generalized greatly and that we should be able to justify the methodology of Césaro arrays under much broader conditions. Let us consider some of the areas where such generalisation seems possible:

(i) To begin with, if we go back to the beginning of section 2.1 where we derived equation 6 giving the contribution of "second-component" pieces as  $\frac{L}{u}$ , we assumed  $\phi$  to be classically integrable on  $[0, \infty)$  and to satisfy equation 2. However, clearly all we require for this aspect of Césaro array methodology is that  $\mathop{Clim}_{X \rightarrow \infty} \int_0^{Xu} \phi(v) dv$  exist and be equal to  $L$ , and this does not require  $\phi(v)$  even to decay to 0 as  $v \rightarrow \infty$  - it could contain divergent components as long as these lead only to Césaro-eigenfunction divergences in  $X$  in this integral.

(ii) As for the later arguments deriving that  $\mathop{Clim}_{X \rightarrow \infty} R(X; u) = \mathcal{S}_0(u)$  in section 2.3, we see that in applying the precise form of the Euler-McLaurin sum formula we needed a number of conditions in order to go, for example, from equation 56 to equation 58. Specifically, we need that  $\mathop{Clim}_{k \rightarrow \infty} \phi(ku) = 0$ ; and that both  $\mathop{Clim}_{X \rightarrow \infty} \int_k^X \phi(\tilde{X}u) d\tilde{X} = 0$  and  $\mathop{Clim}_{X \rightarrow \infty} \int_k^X \check{q}_0(\tilde{X}) \phi'(\tilde{X}u) d\tilde{X} = 0$ ; and that we be able to pass in the limit as  $X \rightarrow \infty$  from  $\int_0^X \check{q}_0(\tilde{X}) \phi'(\tilde{X}u) d\tilde{X}$  to  $\int_0^\infty \check{q}_0(\tilde{X}) \phi'(\tilde{X}u) d\tilde{X}$  in a way which leaves this latter integral susceptible to our subsequent analysis based on the Fourier series for  $\check{q}_0(\tilde{X})$ . While it is somewhat challenging to tease out exactly what these conditions do require, they certainly do not need anything nearly so restrictive as a requirement that  $\phi$  be Schwartzian and indeed, as we saw in section 2.2, they are satisfied by

$\phi(z) = \frac{1}{(z+1)^2}$  from example 3.

(iii) As for this subsequent working using the Fourier series decomposition of  $\check{q}_0(\alpha)$  to analyse  $\int_0^\infty \sum_{n=1}^\infty \frac{\sin(2\pi n \frac{v}{u})}{2\pi n} \phi'(v) dv$  (really  $\mathop{Clim}_{X \rightarrow \infty} \int_0^{ku} \sum_{n=1}^\infty \frac{\sin(2\pi n \frac{v}{u})}{2\pi n} \phi'(v) dv$ ) - or equivalently to analyse  $\int_0^\infty \sum_{n=1}^\infty \frac{i}{2\pi n} \{e^{2\pi i n \frac{v}{u}} - e^{-2\pi i n \frac{v}{u}}\} \phi'(v) dv$  (again really  $\mathop{Clim}_{X \rightarrow \infty} \int_0^{ku} \sum_{n=1}^\infty \frac{i}{2\pi n} \{e^{2\pi i n \frac{v}{u}} - e^{-2\pi i n \frac{v}{u}}\} \phi'(v) dv$ ) - we need first to be able to reverse the order of integration and summation, and then to be able to evaluate the resulting integrals and demonstrate that they give rise to a contribution of  $-\sum_{m \text{ odd}} c_m \zeta(-m) u^m + \mathcal{S}_0(u)$ .

In section 2.3 we used the Schwartzian condition on  $\phi$  in order to guarantee the applicability of the dominated convergence theorem and thereby accomplish the reversal of summation and integration; and then also to tap into existing results regarding Fourier transforms and their properties for such Schwartzian functions. We were able to do so rigorously at least for  $\phi$  even, and to see in outline how the working should proceed in the case of  $\phi$  odd. In the latter case, the working even when  $\phi$  is Schwartzian was not quite fully rigorous, but we believe that only a moderate amount of extra care should be sufficient to render it so. However, relaxing the Schwartzian condition on  $\phi$  and trying to prove further, less restrictive general results guaranteeing the validity of Césaro array methodology, will likely require more delicate analysis and integral estimation than we have needed here. The sketch-proof for  $\phi$  odd which we gave in section 2.3 still gives promise, however, of being the right path to follow, and the fact that we were able, in section 2.2, to demonstrate the required behaviour for the case of  $\phi(z) = \frac{1}{(z+1)^2}$  using explicit integrals from [7] gives strong promise that considerable relaxation should be possible even within the world of classical convergence, let alone the broader framework of Césaro convergence.

### 3 Final remarks

We conclude with three final observations:

(i) In both the examples from section 2.2, and in the general working from section 2.3, the engine driving our deduction that  $\mathop{Clim}_{X \rightarrow \infty} R(X; u) = \mathop{Clim}_{X \rightarrow \infty} \phi((X - \check{q})u)$  was in  $\mathcal{S}_0(u)$  in each case was the precise form of the Euler-McLaurin sum formula, combined with subsequent analysis of its integral "error term". In some sense, then, we can think of Césaro array methodology as representing the marriage of degree-wise generalised Césaro methods with the analytical power of the Euler-McLaurin sum formula to obtain a new tool in the generally delicate business of asymptotic analysis.<sup>8</sup>

Moreover, note that it was (a) quite difficult, in the treatment of specific examples in section 2.2 and even more so in general in section 2.3, to demon-

<sup>8</sup>As such a tool, it augments existing methods such as the stationary phase approximation, its extension using Feynman diagrams, and so on.

strate the Schwartzian character of  $\mathop{Clim}_{X \rightarrow \infty} \phi((X - \check{q})u)$ ; and yet (b) in all cases  $\mathop{Clim}_{X \rightarrow \infty} \phi((X - \check{q})u)$  was indeed Schwartzian, even where  $\phi$  itself was not. Together, these observations suggest that the Césaro array methodology may indeed capture something quite subtle re the apparent Schwartzian nature of functions of the form  $\mathop{Clim}_{X \rightarrow \infty} \phi((X - \check{q})u)$  in general. If a more general result along these lines could be established - at least a much broader one than we have deduced in Result 1 - it would thus, in our opinion, represent a valuable advance. This would be especially so, since whenever this is the case, the Césaro array methodology gives us the asymptotics of  $\sum_{j=1}^{\infty} \phi(ju)$  as  $u \rightarrow 0^+$  so simply and directly (equation 7). We believe that at least some significant extension of Result 1 should, in fact, be not too difficult to achieve for mathematical road-builders with the requisite analysis tool kit, and we hope there are some who are interested enough to take up the challenge.

(ii) Of course some generalization of reach is possible without even obtaining any extension of Result 1. Equation 7 only ever entails a  $\frac{1}{u}$ -style singularity as  $u \rightarrow 0^+$ , but higher-order poles at 0, and indeed any desired Laurent-series singular piece, are readily obtainable without any further effort. For example, if we instead consider  $\sum_{j=1}^{\infty} \phi(ju^n)$  then our treatment of the second-component pieces yields immediately a singular term of  $\frac{L}{u^n}$  (invoking Césaro dilation-invariance here if we have needed to perform a Césaro evaluation of the relevant integral arising in equation 6); and by combining such series for different  $n$  we could clearly obtain any Laurent singular-component we desire. In turn, by considering sums of the form  $\sum_{j=1}^{\infty} \phi(jh(u))$  where  $h(u)$  is a polynomial (or perhaps even a more general function) we could get more complicated asymptotic outcomes from our Césaro array methodology without necessarily needing to extend the existing state of our machinery.

(iii) Finally we note, along these lines, that Césaro array methodology can be extended further in many directions. For example, we need not simply consider a series of the form  $\sum_{j=1}^{\infty} \phi(jh(u))$  for a single choice of function  $\phi$  and  $h$ . Alternatively we could require only generalised Césaro convergence, whereas in all the examples we have considered so far we have in fact had classical convergence; and so on. In fact, one of the real virtues of Césaro array methodology, in our opinion, is that, as articulated in [IV], it is a methodology for asymptotic analysis which is both very simple (witness the the three primary examples we have considered) and very flexible. It requires only that we be able to formally reverse order of summation and then be able to handle both the resulting second-component integral and the generalised Césaro analysis of the degree-wise p-sums for the first-component pieces.

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