Generalised Césaro Convergence And Its Applications - A Selected Survey

Richard Stone

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Abstract

This is a brief survey covering twelve "highlights" from a collection of four sets of papers on generalised Césaro convergence and its applications. The four sets are "Introduction to generalised Césaro convergence I-III", "Césaro Arrays I-III", "Root Identities I" and "Root Identities for ζ IIA-IIC", and finally "Taylor Series to the Left I-V". For reasons canvassed in a single-page apologia at the outset[1], it seems unfeasible[2] to seek publication of these papers within existing journals, at least initially, but this is not because of mathematical unsuitability or lack of either rigour or interest. As such, the purpose of this paper is simply to outline 12 highlights from the four sets, in the hope of convincing readers to take an interest in these papers, and in the methods and results developed in them. If sufficient interest is generated - in particular if sufficient support for doing so were to be gained - then publication in "proper" journals could be undertaken. The twelve highlights cover what we hope are a range of claims and methods of clear interest; and while we only state results here (leaving all proof and detail to the papers themselves), the papers are all published on the ArXiv, where such detail can be thoroughly assessed. In many cases, stated results can also be independently checked (e.g. numerically) as a means of directly testing the trustworthiness of the methods and results.[3]

1 Three Highlights from Introduction to Generalised Césaro Convergence I-III

[Highlight 1 - Generalised Césaro Convergence]: We let P and P_D be the continuous ($P[f](X) := \frac{1}{X}\int_0^X f(x)\,dx$ for functions) and discrete ($P_D[\{a_j\}]_k = \frac{1}{k}\sum_{j=1}^k a_j$ for sequences) versions of the Césaro averaging operator. By considering not just pure powers (P^N or P_D^N representing repeated averaging), but

[1]Introduction to Generalised Césaro Convergence I, section 1.2

[2]to the author at any rate

[3]We also note here that in the papers themselves we aim to write in as accessible a manner as possible, and to be as straightforward as we can be in regards to areas where there are gaps to fill for proofs or methods.

regular polynomials ($q(P)$ or $q(P_D)$ with $q(1) = 1$) we extend from traditional to generalised Césaro convergence.

- This allows us to assign limit 0 to (i.e. annihilate) eigenfunctions and generalised eigenfunctions of P (i.e. power and power-log divergences), and hence to assign generalised limits to a much greater range of functions (or in the discrete case sequences). Applied to divergent sums, it allows us to assign a generalised Césaro value to a much broader class of divergent series. In turn this allows us to undertake concrete analytic continuation in a much greater range of cases.
  - For example, working in the continuous Césaro framework, we perform the constructive analytic continuation of ζ based on its defining series ($\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$) from its domain of classical convergence on $Re(s) > 1$ to all of $\mathbb{C}$ via generalised Césaro summation. This includes identifying poles at s-values where the partial sum function develops a pure log-divergence (since these represent eigenfunctions and generalised eigenfunctions with eigenvalue 1, to which no Césaro limit can be attached), as well as calculating the residues at such poles using generalised Césaro L'Hopital's calculations. In the same manner, we identify when generalised Césaro methods lead to removable singularities or "anomalies" and how the correct analytically-continued value can be restored by similar means.
- The key to this working is the derivation of the core generalised Césaro result that, if $X = k + \alpha$ where $k \in \mathbb{Z}$ and $\alpha \in [0, 1)$ then

$$\sum_{j=1}^{k} j^{-s} = \frac{X^{1-s}}{1-s} + \zeta(s) - R(X) \tag{1}$$

  where $R(X)$ converges in a *strong* Césaro sense to 0 as $X \to \infty$ (meaning that there is a pure power P^N - in this case $N = Floor(Re(-s)) + 1$ - such that $P^N[R](X) \to 0$ classically).
- Related interesting results which we prove regarding generalised Césaro convergence include that, for $\delta \in \mathbb{C}$, $Re(\delta) > 0$, and $r \in \mathbb{Z}_{\geq 0}$:

$$k^{\delta}\alpha^{r} \overset{C}{\to} \begin{cases} 0 & if\ \delta \notin \mathbb{Z}_{\geq 0} \\ \frac{(-1)^n}{n+r+1} & if\ \delta = n \in \mathbb{Z}_{\geq 0} \end{cases} \tag{2}$$

  or equivalently

$$X^{\delta}\alpha^{r} \overset{C}{\to} \begin{cases} 0 & if\ \delta \neq 0 \\ \frac{1}{r+1} & if\ \delta = 0 \end{cases} \tag{3}$$

  and that

$$\int_{-1}^{0} \sum_{j=1}^{k} j^{n} \, \mathrm{d}k = \zeta(-n) \quad \text{for all } n \in \mathbb{Z}_{\geq 0} \; . \tag{4}$$

[Highlight 2 - Geometric Generalised Césaro Convergence, Remainder Summation and Applications]: We extend generalised Césaro convergence to encompass convergence along a contour in $\mathbb{C}$ tending to ∞. Most commonly this will be a horizontal ray originating at some point z_0, but it may also be a vertical ray (as, for example, when considering sums of non-trivial roots of ζ up or down the critical line), or indeed any contour. The most basic application is for "remainder summation" analysing the convergence of a p-sum function[4] of a function, f, starting from a point $z_0 \in \mathbb{C}$, i.e. $R_+[f](z_0) = \sum_{j=1}^{\infty} f(z_0 + j)$.

- Critically, it turns out that *geometry* now becomes crucial. It is eigenfunctions and generalised eigenfunctions in the geometric variable $z = \gamma(t)$ on contour γ (e.g. $z = z_0 + t = z_0 + k + \alpha$ for remainder summation) which have generalised Césaro limit 0; thus it is power and power-log functions in z, not t, which we may "annihilate" in applying generalised geometric Césaro convergence; and so the *location* of summands now becomes important, not just the summands themselves.

- Remainder summation allows extension of finite sums from a discrete variable k ($\sum_{j=1}^{k} a_j$) to a continuous variable $z_0 \in \mathbb{C}$ as a difference of two geometric generalised Césaro remainder sums:

$$\sum_{j=1}^{z_0} f(j) := R_+[f](0) - R_+[f](z_0) \tag{5}$$

  This allows calculus (differentiation, integration etc) to be applied to such finite sums (as in equation 4). Finite products can be extended in the same way via exponentiation of logarithms.

  - For example, $\sum_{j=1}^{z_0} 1 = -z_0 - \frac{1}{2}$ and the Hurewicz zeta function is given by $\zeta_H(z_0; s) = R_+[\tilde{z}^{-s}](z_0)$.
  - Likewise $\Gamma(z_0 + 1) = \prod_{j=1}^{z_0} j$ and this represents the most natural possible way of defining $\Gamma(z_0 + 1)$ as the extension of $k!$ from $k \in \mathbb{Z}$ to arbitrary $z_0 \in \mathbb{C}$.
  - The criticality of geometry in these definitions is perfectly illustrated by the case of $\Gamma(z_0 + 1)$. It is crucial in ensuring that $R_+[\ln \tilde{z}](z_0)$ is well-defined in a generalised Césaro sense (no pure log-divergences!); and it is vital in resolving the apparent problem with log-divergence in calculating $\Gamma'(1)$ and arriving at the correct "renormalized" value of $\Gamma'(1)$ as $-\gamma$.
  - The functional equation for Γ (that $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$) follows from a geometric argument involving bi-directional summation ($R_{+,0,-}$) and its natural connection with Fourier theory. The argument proceeds in a way which can be replicated for many other functions (not just $\ln z$) and which thus explains why many functional equations relate function values at z and $1 - z$.

[4] partial-sum function

- We also prove that generalised geometric Césaro convergence/summation is both dilation-invariant and scaling-invariant:
  - The usually mysterious multiplication formulae for $\Gamma(z+1)$ then become a natural set of formulae which follow trivially from Césaro dilation-invariance combined with a simple geometric figure.
  - Equation 4 can be subsumed within a more general result for arbitrary $s \in \mathbb{C} \setminus \{1\}$, not just $n \in \mathbb{Z}_{\geq 0}$, namely that

$$\int_{-1}^{0} \zeta_H(z_0; s)\, dz_0 = 0 \tag{6}$$

    and this is again easily proved using the dilation-invariance of generalised geometric Césaro remainder summation.

[Highlight 3 - Césaro-adapted scales and formal calculus]: We introduce the key concept of a Césaro-adapted scale - a ladder of functions, each of which is an anti-derivative of the preceding one and each of which converges to 0 in a strong Césaro sense under a single application of P. For example, the ladder of rescaled periodic Bernoulli polynomials $\overset{\vee}{q}_0(X) = \alpha - \frac{1}{2}$, $\overset{\vee}{q}_1(X) = \frac{1}{2}\alpha^2 - \frac{1}{2}\alpha + \frac{1}{12}$, $\overset{\vee}{q}_2(X) = \frac{1}{3}\alpha^3 - \frac{1}{2}\alpha^2 + \frac{1}{6}\alpha$, $\overset{\vee}{q}_3(X) = \frac{1}{4}\alpha^4 - \frac{1}{2}\alpha^3 + \frac{1}{4}\alpha^2 - \frac{1}{120}, \ldots$ form an important example of such a Césaro-adapted scale (here $X = k + \alpha$ as always).

- This concept, and in particular the example of the $\left\{\overset{\vee}{q}_n(X)\right\}_{n=0}^{\infty}$, allows rigorous proof of the core results in equations 1, 2 and 3 and in turn of all the subsequent results thus far. Césaro-adapted scales also play a critical role in later explicit calculations of certain generalised geometric Césaro sums over non-trivial roots of ζ.

- Calculations with Césaro-adapted scales, and associated generalised Césaro calculations, are well-adapted to introducing formal symbols (e.g. the formal symbol τ such that $\tau^s := \zeta(-s)$) and formal function elements (e.g. $\overset{\vee}{q}$ such that $\overset{\vee}{q}^n := \overset{\vee}{q}_n(\alpha)$).These allow many of the quantities we are interested in, and results we wish to prove, to be expressed in very simple form:
  - For example, $\overset{\vee}{q}^n := \overset{\vee}{q}_n(\alpha) = (\alpha - \tau)^n$ and this formulation allows us to extend to $\overset{\vee}{q}_\rho(\alpha)$ as $(\alpha - \tau)^\rho$ for arbitrary $\rho \in \mathbb{C}$; likewise the residual function $R(X)$ in equation 1 is given very simply as $R(X) = (X - \overset{\vee}{q})^{-s}$ and even a result like the Euler-McLaurin sum formula can be re-expressed extremely compactly as

$$\sum_{j=1}^{k} f(j) = C_f - f(k+\tau) \tag{7}$$

- Calculation in the formal jungles is powerful, but somewhat mysterious and care needs to be taken to observe certain rules - no evaluation of formal quantities "until the end"; retain "stand-alone zeros", since they may be brought back to life by interaction with other formal quantities before evaluation; consider power series indices as extending both to the left and to the right and embed them within a continuous "fabric"; take care with development of delta-functions in final evaluation etc.

- Provided we are careful, however, calculus with formal quantities - including differentiation and integration as well as algebra and combinatorics - can be both simplifying and powerful:

  – For example, we have $\zeta_H(z_0; s) = (z_0+\tau)^{-s}$, and thus ζ_H is connected to the $\overset{\vee}{q}_\rho$ by $\overset{\vee}{q}_\rho(\alpha) = e^{i\pi\rho}\zeta_H(-\alpha; -\rho)$; at the same time we have the useful identity that $\tau^{-s} = (\tau-1)^{-s}$ for all $s \in \mathbb{C} \setminus \mathbb{Z}_{>0}$.

  – We can use Fourier theory on the $\overset{\vee}{q}_\rho(\alpha)$ and hence derive the functional equation for ζ by specialising to $\alpha = 0$.

  – By our compact re-expression of the Euler-McLaurin sum formula we have that $\sum_{j=1}^{k} j^{-s} = \zeta(s) - (k+\tau)^{-s}$ and this leads to a very simple derivation of our core result equation 1.

  – We get a completely trivial (effectively one line) proof of our earlier result that $\int_{-1}^{0} \zeta_H(z_0; s)\, dz_0 = 0$ and we find that $\frac{d}{dX}(X - \overset{\vee}{q})^n = X^n \cdot \left\{1 - \sum_{j=1}^{\infty} \delta_j(X)\right\}$.

  – We get a number of other fascinating observations/speculations.

2 Two Highlights from Césaro Arrays I-III

[Highlight 4 - The method of Césaro Arrays]: The method of Césaro arrays allow us to successfully reverse order of summation and obtain a power series expansion for $f(u) := \sum_{j=1}^{\infty} f_j(u)$ just from a knowledge of the power series expansions of the f_j:

- Seamlessly reversing order of summation in this way is obviously desirable, but usually can't be done under classical convergence. However, the method of Césaro arrays makes it possible.

- For example, we can derive directly that

$$\sum_{j=1}^{\infty} e^{-jz} = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\zeta(-n)z^n \tag{8}$$

  and deduce the highly non-trivial power series expansions that

$$H(z) := \frac{1}{2} + \sum_{j=1}^{\infty} e^{-\pi j^2 z^2} = \frac{1}{2}\frac{1}{z} + \mathcal{S}_0(z) \tag{9}$$

and that

$$-\frac{1}{x}+\sum_{j=1}^{\infty}\frac{1}{(jx+1)^2}=\sum_{j=0}^{\infty}B_{j+1}x^j+\mathcal{S}_0(x) \quad . \tag{10}$$

- The methodology **(a)** uses detailed Césaro convergence to sum divergent series at each degree (giving component-1 pieces); **(b)** sequesters the eigenfunction-divergences and re-sums them separately to obtain the singular terms which arise (the component-2 pieces - these are often at degrees which don't even appear in the power series for the individual $f_j(u)$!); and **(c)** uses *strong* Césaro convergence in the residual pieces to leave only an overall Schwartzian residual. It can be applied within the continuous or discrete Césaro framework.

[Highlight 5 - Boundary singularity structure of $H(z)=\frac{1}{2}+\sum_{j=1}^{\infty}e^{-\pi j^2z^2}$; Exponential sums; General sums $\sum_{j=1}^{\infty}a_jj^r$ where $\{a_j\}$ is periodic]: The structure of singularities for $H(z)$ on the boundary $Im(\ln z)=\pm\frac{\pi}{4}$ is extraordinary (and plays a key role in Hardy's proof that there are infinitely many non-trivial zeros of ζ on the critical line). By considering on the one hand 1-d generalised Césaro summation in concert with the functional equation for $H(z)$ and dilation-invariance, and on the other 2-d Césaro arrays, we are able to deduce much of this structure and its close linkage to exponential sums. This includes evaluation of some famous exponential sums by comparison of singular terms. Each of these exponential sums attached to a singular piece is just the first in a countable collection of related Césaro sums we can also read off. The 2-d Césaro array calculations give rise to general consideration of sums of the form $\sum_{j=1}^{\infty}a_jj^r$ where a_j is a periodic sequence, and we give a mixture of general results and intriguing conjectures for such sums (some including formal quantities).

- For $\nu_{2,p}=\sqrt{\frac{-2i}{p}}$ on the lower boundary (p odd), if we approach $\nu_{2,p}$ at right-angles (i.e. $z^2=\frac{-2i}{p}+\epsilon$ and $\epsilon\to 0^+$) then

$$H(z)=\begin{cases}\frac{1}{2\sqrt{p}}\frac{1}{\sqrt{\epsilon}}+\mathcal{S}_0(\epsilon) & \text{if} \quad p\equiv 1\ (mod\ 4)\\ \\ \frac{i}{2\sqrt{p}}\frac{1}{\sqrt{\epsilon}}+\mathcal{S}_0(\epsilon) & \text{if} \quad p\equiv 3\ (mod\ 4)\,.\end{cases} \tag{11}$$

  - We prove this using the functional equation that $H(z)=\frac{1}{z}H\left(\frac{1}{z}\right)$ and the dilation-invariance of generalised Césaro summation.

- Likewise for $\nu_{1,p}=\sqrt{\frac{-i}{p}}$ we get that

$$H(z)=\begin{cases}\frac{e^{i\frac{\pi}{4}}}{2\sqrt{p}}\frac{1}{\sqrt{\epsilon}}+\mathcal{S}_0(\epsilon) & \text{if } p \text{ even}\\ \\ \mathcal{S}_0(\epsilon) & \text{if } p \text{ odd}\,.\end{cases} \tag{12}$$

- Hence branch-point singularities and Schwartzian zeros interleave densely on the boundary of $H(z)$ in any neighbourhood of 0.

- We can also consider these cases using 2-d Césaro arrays for $\sum_{j=0}^{\infty} a_j e^{-\pi j^2 \epsilon}$ where a_j is a periodic sequence (e.g. $a_j = e^{2\pi i \frac{j^2}{p}}$ or $a_j = e^{\pi i k \frac{j^2}{p}}$ more generally). Harnessing Césaro-adapted scales within this Césaro-array analysis allows us to deduce certain famous exponential sums just from comparison of the singular term with that derived earlier, e.g. Gauss's sum:

$$\sum_{j=0}^{p-1} e^{2\pi i \frac{j^2}{p}} = \sigma_{2,p} = \begin{cases} \sqrt{p} & , \ p \equiv 1 \ (mod\ 4) \\ \\ i\sqrt{p} & , \ p \equiv 3 \ (mod\ 4) \, . \end{cases} \tag{13}$$

and

$$\sum_{j=0}^{p-1} e^{i\pi \frac{j^2}{p}} = \sigma_{1,p} = \begin{cases} e^{i\frac{\pi}{4}} \cdot \sqrt{p} & , \ p \text{ even} \\ \\ 0 & , \ p \text{ odd} \, . \end{cases} \tag{14}$$

- By comparison with the non-singular degrees, we also get a countable collection of additional discrete Césaro sums in each case: e.g. for p odd, we get that

$$\sum_{j=1}^{\infty} j^{2n} \left(e^{2\pi i \frac{j^2}{p}} - \frac{\sigma_{2,p}}{p} \right) = 0 \tag{15}$$

strongly via P_D^{2n+1} where $\sigma_{2,p}$ is as defined above; and similarly

$$\sum_{j=1}^{\infty} j^{2n} \left(e^{i\pi \frac{j^2}{p}} - \frac{\sigma_{1,p}}{p} \right) = 0 \quad . \tag{16}$$

- For $\nu_{k,p} = \sqrt{\frac{-ik}{p}}$ (where trying to use the functional equation for $H(z)$ and 1-d Césaro methods fails), 2-d Césaro array analysis still works and gives dense interleaving of branch-point singularities and Schwartzian zeros all along the convergence boundary of $H(z)$, with the character of each $\nu_{k,p}$ being determined by whether the finite exponential sum $\sum_{j=0}^{N-1} e^{i\pi k \frac{j^2}{p}}$ is zero or not (where $N = p$ or $N = 2p$ depending on k and p).

- In general, we can also naturally extend to consider $\sum_{j=0}^{\infty} j^{2n} \cdot \tilde{a}_j$ where $\{\tilde{a}_j\} \in \mathcal{S}_{[N]}$ is strongly-periodic (meaning it is period-N and $\sum_{j=0}^{N-1} \tilde{a}_j = 0$) and satisfies $\tilde{a}_{N-j} = \tilde{a}_j$. In this setting we can prove that $\sum_{j=0}^{\infty} j^{2n} \cdot \tilde{a}_j = 0$ for $n = 1, 2, 3$ and we conjecture that this is true in general. Proving this, however, is challenging and leads already to some very interesting results/conjectures combining combinatorics and formal quantities.

3 Two Highlights from Root Identities I and Root Identities for ζ IIA-IIC

[Highlight 6 - Generalised Root Identities - Meaning and Non-Trivial Examples]: We introduce a new set of generalised root identities which state that

$$-\frac{1}{\Gamma(\mu)}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{\mu}(\ln(f(z))\Big|_{z=z_0} = e^{i\pi\mu}\cdot\sum_{\{roots\, r_i\}}\frac{M_i}{(z_0-r_i)^{\mu}} \tag{17}$$

for arbitrary $z_0, \mu \in \mathbb{C}$. These generalise basic root identities for polynomials to broader classes of functions. They also generalise Hadamard's product formula (which, where applicable, corresponds to the case of $\mu = 1$) and its derivatives ($\mu \in \mathbb{Z}_{\geq 2}$) to non-entire functions and arbitrary $\mu \in \mathbb{C}$.

- Even for $\mu \in \mathbb{Z}_{\geq 1}$, the generalised root identities are only true for one representative function within a natural equivalence class of functions - this representative can be found by successively removing obstructions. In practice, however, they are true for many interesting natural functions such as $\cos\left(\frac{\pi z}{2}\right)$, $\Gamma(z+1)$, $\pi^{-\frac{s}{2}}\zeta(s)$ and therefore $\xi(s)$ (although again we demonstrate this initially only for $\mu \in \mathbb{Z}_{\geq 1}$).

- As full identities, both sides need further explanation. The RHS is the "root side", $r_f(z_0, \mu)$, and needs to be interpreted in a generalised geometric Césaro sense whenever the sum is divergent. The "roots" are generalised roots, including ordinary roots, poles and even branch points. Geometry is *crucial* - the term $\frac{M_i}{(z_0-r_i)^{\mu}}$ must be added in at $z_0 - r_i$ or the identity fails. Thus, as z_0 varies, not only the summands, but also their locations, vary!

- The LHS is the "derivative side", $d_f(z_0, \mu)$ and, for $\mu \notin \mathbb{Z}_{>0}$, must be interpreted using Fourier theory (cf pseudo-differential operators) or fractional calculus; it at least satisfies that

$$\left(\frac{d}{dz}\right)^{\mu}(a^z)\Big|_{z=z_0} = a^{z_0}\cdot(\ln a)^{\mu}$$

and

$$\frac{1}{\Gamma(\mu+1)}\left(\frac{d}{dz}\right)^{\mu}(z^{\rho})\Big|_{z=0} = \begin{cases}1 & , \rho=\mu \\ 0 & , \text{else.}\end{cases}$$

- When $\mu \notin \mathbb{Z}_{>0}$, $\left(\frac{d}{dz}\right)^{\mu}$ is non-local and the generalised root identities for $\mu \in \mathbb{Z}_{\leq 0}$ are of particular interest (and not always zero despite the singularity of $\Gamma(\mu)$). Distributional results come into play and the generalised root identities give identities in which the asymptotic behaviour of generalised roots as they approach ∞, and their geometric location, become critical. We thus gain information about these asymptotics and geometric locations.

- Examples:
  - $f(z) = z$ satisfies the generalised root identities but this is non-trivial (equivalent to proving the generalised Fresnel integrals for $\sin(x^a)$ and $\cos(x^a)$); hence polynomials satisfy them also.
  - $f(z) = \cos\left(\frac{\pi z}{2}\right)$ - the fact this satisfies the generalised root identities for arbitrary $\mu \in \mathbb{C}$ (not just $\mu \in \mathbb{Z}_{\geq 1}$) is equivalent to proving the functional equation for ζ (i.e. $\zeta(1-\mu) = 2^{1-\mu}\pi^{-\mu}\cos\left(\frac{\pi\mu}{2}\right)\Gamma(\mu)\zeta(\mu)$)
  - $f(z) = \Gamma(z+1)$ satisfies the generalised root identities. We use the Euler-McLaurin sum formula to make sense of $r_\Gamma(z_0, \mu)$ and Fourier and distribution theory to get an explicit formula for $d_\Gamma(z_0, \mu)$. These allow us to read off equality for $\mu = -n$ and to easily demonstrate it numerically for $\mu \in \mathbb{C} \setminus \mathbb{Z}$. The case of $\mu \notin \mathbb{Z}_{\leq 0}$ can be tightened from numerical demonstration to full proof. Moreover, using the generalised root identities for $\mu = 0, -1, -2, \ldots$ successively, we can deduce Stirling's theorem for Γ, showing how the identities for $\mu \in \mathbb{Z}_{\leq 0}$ do indeed give information about the asymptotics and geometric location of the roots of a function.

[Highlight 7 - The generalised root identities for ζ and a new family of integral identities for the argument of the zeta function, $S(T)$]: Using fractional calculus, we can show that, apart from the trivial obstruction at $\mu = 1$ (where we need to consider instead $\pi^{-\frac{s}{2}}\zeta(s)$), ζ does satisfy the generalised root identities for all $\mu \in \mathbb{C} \setminus \{1\}$ and $Re(s_0) > 1$. This can be confirmed numerically using e.g. the first $10,000$ primes, 2x10^6 trivial (T) roots and $2,000,052$ non-trivial (NT) roots.

- The Euler product formula gives that

$$d_\zeta(s_0, \mu) = -\frac{e^{i\pi\mu}}{\Gamma(\mu)} \sum_{p\,prime} (\ln p)^\mu \sum_{n=1}^{\infty} n^{\mu-1}\, p^{-ns_0} \quad . \tag{18}$$

  As for $r_\zeta(s_0, \mu)$, it has contributions from the pole at $s = 1$, the trivial roots T and the non-trivial roots $NT = NT_+ \cup NT_-$:

$$r_\zeta(s_0, \mu) = r_P(s_0, \mu) + r_T(s_0, \mu) + r_{NT}(s_0, \mu) \quad . \tag{19}$$

- If we consider the cases of $\mu = -n \in \mathbb{Z}_{\leq 0}$, then $d_\zeta(s_0, -n) = 0$ for all $Re(s_0) > 1$ (because in this case the singularity of $\Gamma(-n)$ is not offset by $\sum_{p\,prime}$, which remains bounded). If we assume the Riemann hypothesis (RH), then we can also try to explicitly calculate the $r_{NT}(s_0, -n)$ component, and hence also $r_\zeta(s_0, -n)$ as a whole. To do this we use the Riemann-von Mangoldt formula for $N(T) = \sum_{0<Im(\rho_i)<T} 1$. This says that

$$N(T) = \check{N}(T) + S(T) + \frac{1}{\pi}\delta(T) \tag{20}$$

where

$$\check{N}(T) = \frac{T}{2\pi}\ln(\frac{T}{2\pi}) - \frac{T}{2\pi} + \frac{7}{8} \tag{21}$$

and $S(T)$ is the famous argument of the ζ-function, and

$$\delta(T) = \frac{T}{4}\ln\left(1+\frac{1}{4T^2}\right) + \frac{1}{4}\tan^{-1}\left(\frac{1}{2T}\right) + \frac{T}{2}\int_0^\infty \frac{\overset{\vee}{q}_0(u)}{(u+\frac{1}{4})^2+(\frac{T}{2})^2}\mathrm{d}u\ . \tag{22}$$

As long as we are very careful to respect the geometric placement of summands, we *can* in fact perform this evaluation of the $r_{NT}(s_0,-n)$ (and hence $r_\zeta(s_0,-n)$) as generalised Césaro sums. Dilation-invariance also plays a key role in these computations, along with the use of Césaro-adapted scales to handle both the terms involving $S(T)$ and the integrals involving $\overset{\vee}{q}_0(u)$ in the terms involving $\delta(T)$.

- On doing these (lengthy) calculations and then setting $r_\zeta(s_0,-n) = 0 = d_\zeta(s_0,-n)$, we finally obtain that:
  - For n odd, no new information is gained (i.e. we confirm that $r_\zeta(s_0,-n) = 0 = d_\zeta(s_0,-n)$ unconditionally), but;
  - For $n = 2k$ even, the fact that $r_\zeta(s_0,-2k) = 0$ for all $k \in \mathbb{Z}_{\geq 1}$ implies a new family of strong Césaro integral identities which $S(T)$ must satisfy conditional on RH, namely that

$$\int_0^\infty t^{2k}\mathrm{d}S(t) \;=\; (-1)^k\frac{1}{2^{2k}} \quad . \tag{23}$$

  - We may express these identities alternatively as saying that

$$\int_{-\infty}^\infty t^n\,\mathrm{d}S(t) = \begin{cases} 0 & n = 2k-1 \quad \text{odd} \\ (-1)^k\frac{1}{2^{2k-1}} & n = 2k \quad \text{even.} \end{cases} \tag{24}$$

  - This means that $\int_0^T t^{2k}\mathrm{d}S(t) \overset{C}{\to} (-1)^k\frac{1}{2^{2k}}$ in a strong Césaro sense as $T\to\infty$ via the pure power P^{2k+1} (i.e. via $(2k+1)$-fold averaging).
  - This result can be recast in terms of the iterated integrals $S_1(T) := \int_0^T S(t)dt$, $S_2(T) := \int_0^T S_1(t)dt$ etc, but in either case it is numerically testable (for those who remain skeptical despite the proof!)

- These are a family of completely new integral identities for $S(T)$ conditional on RH, and the fact that they are strong Césaro identities makes them perfectly adapted for using Césaro arrays to obtain the asymptotics of general integrals of the form $\int_0^\infty f(ut)\,dS(t)$ or $\int_{-\infty}^\infty f(ut)\,dS(t)$:
  - For example if f is an even function in a neighbourhood of 0 then we get that

$$\int_0^\infty f(ut)\,dS(t) = f\left(\frac{iu}{2}\right) + \mathcal{S}_0(u) \tag{25}$$

asymptotically as $u \to 0$ and this would be interesting to test numerically for suitable choices of f. Since the closer u gets to 0 the more relevant the behaviour of $S(t)$ for large t becomes, it would also be interesting to consider what might possibly be deduced re the structure and distribution of NT-zeros by a wise choice of "test function" f.

 - For arbitrary f analytic near 0 (not necessarily even), we can alternatively conclude that

$$\int_{-\infty}^{\infty} f(ut)\, dS(t) = \left\{ f\left(\frac{iu}{2}\right) + f\left(-\frac{iu}{2}\right) \right\} + \mathcal{S}_0(u) \tag{26}$$

   as $u \to 0$. Thus e.g. the Fourier transform of $S'(t)$ is given asymptotically by

$$\mathcal{F}[S'(t)](u) = \int_{-\infty}^{\infty} e^{-itu}\, dS(t) = \left\{ e^{\frac{u}{2}} + e^{-\frac{u}{2}} \right\} + \mathcal{S}_0(u) \tag{27}$$

   as $u \to 0$, which seems interesting ...; and again other choices of "test function" f seem worth considering.

- Separately, the calculations of $r_{NT}(s_0, -n)$ are independently interesting in showing the criticality of geometry in the placement of NT-roots in order for the generalised root identities to hold (which they do!). Other natural placements of roots (e.g. placement of ρ_i at ρ_i rather than at $z_0 - \rho_i$ in the complex plane) also lead to elegant, simple formulae for sums like $\sum_{\rho_i \in NT} \rho_i^n$, at least modulo RH again - and these seem independently interesting, even though they do not arise in the context of the generalised root identities. Indeed they arise naturally in trying to use Césaro arrays to understand functions like $\psi(X)$ via von Mangoldt's theorem. This is another avenue to explore (and one where progress has been made, albeit some stumbling blocks remain ...).

4 Five Highlights from Taylor series to the left I-V

[Highlight 8 - Taylor series to the left and integration]: The Taylor series $\sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} x^j$ of f around 0 can be extended "to the left" with extra terms

$$\frac{f^{(-1)}(0)}{(-1)!} \cdot \frac{1}{x} + \frac{f^{(-2)}(0)}{(-2)!} \cdot \frac{1}{x^2} + \dots \quad . \tag{28}$$

Morally, we should have

$$f^{(-1)}(0) = -\int_0^{\infty} f(x)\, dx \quad \text{and} \quad f^{(-n)}(0) = (-1)^n \cdot F_n(0) \tag{29}$$

where $F_n(x)$ is the n^{th} iterated integral given by

$$F_n(x) = \int_x^\infty F_{n-1}(t)\,dt \quad \text{and} \quad F_0(x) = f(x) \quad . \tag{30}$$

- If we embed the coefficients $\frac{f^{(j)}(0)}{j!}$, $j \in \mathbb{Z}$, "correctly" within a Taylor-coefficient *function* $\overset{\vee}{f}(s)$, $s \in \mathbb{C}$, then we get useful results. For example, if f is integrable on $[0, \infty)$, then $\overset{\vee}{f}(-1) = 0$ and

$$\int_0^\infty f(x)\,dx \;=\; -\overset{\vee}{f}{}'(-1) \;=\; -\lim_{\epsilon \to 0} \frac{\overset{\vee}{f}(-1+\epsilon)}{\epsilon} \quad , \tag{31}$$

  with similar results for $F_n(0)$ and for $\int_0^\infty x^n\, f(x)\,dx$.

- Finding a canonical choice for $\overset{\vee}{f}(s)$ from among the range of gauge equivalent choices so that these integral results are true requires care. The choice is often, however, natural and this allows us to calculate many definite integrals - including ones that are non-trivial by traditional means - by very quick, simple calculation of $-\overset{\vee}{f}{}'(-1)$, generally without doing anything that looks like traditional integration!

- For example, we can trivially calculate that $\int_0^\infty e^{-x^n}\,dx = \left(\frac{1}{n}\right)!$ and $\int_0^\infty \frac{1}{1+x^n}\,dx = \left(\frac{\pi}{n}\right) \cdot \csc\left(\frac{\pi}{n}\right)$; and Dirichlet's integral ($\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$) and the generalised Fresnel integrals ($\int_0^\infty \sin(x^n)\,dx = \left(\frac{1}{n}\right)! \cdot \sin\left(\frac{\pi}{2n}\right)$ and $\int_0^\infty \cos(x^n)\,dx = \left(\frac{1}{n}\right)! \cdot \cos\left(\frac{\pi}{2n}\right)$); and we can calculate quickly that, e.g.

$$\int_0^\infty \left(exp\left(\frac{-1}{1+x^k}\right) - 1 \right) dx = -\frac{\frac{\pi}{k}}{\sin\left(\frac{\pi}{k}\right)} \cdot \left\{ \begin{array}{c} \frac{1}{1!}\binom{\frac{1}{k}-1}{0} + \frac{1}{2!}\binom{\frac{1}{k}-1}{1} \\ \\ +\frac{1}{3!}\binom{\frac{1}{k}-1}{2} + \ldots \end{array} \right\} \tag{32}$$

  and so on.

  - For the last integral in particular, it is not at all obvious how to calculate this by traditional means; but its calculation by TS-to-the-left methods is very quick, and uses nothing more than elementary combinatorics and algebra, together with the use of a formal symbol f for the factorial function (i.e. $f^s := s!$).

  - The other examples use natural forms for the relevant $\overset{\vee}{f}(s)$ functions, directly inferred from the form of the Taylor-coefficients of f, together with a re-scaling rule that if $g(x) := f(x^n)$ then

$$\overset{\vee}{g}(s) = \frac{1}{n} \cdot \frac{\sin(2\pi s)}{\sin(\frac{2\pi s}{n})} \cdot \overset{\vee}{f}(\frac{s}{n}) \quad . \tag{33}$$

- The method of calculating $\int_0^\infty f(x)\,dx$ as $-\overset{\vee}{f}'(-1)$ works irrespective of whether the power series from which $\overset{\vee}{f}(s)$ is inferred has $R = \infty$, finite R or even $R = 0$; so TS-to-the-left methods achieve global calculation from local information (local-to-global inference)!

- In fact, all these results and methods continue to hold even if $\int_0^\infty$ is just a generalised Césaro integral; and f may have poles, or power series expansions which are not just Taylor, but Laurent or even asymptotic series:

  - For this reason we call $\overset{\vee}{f}(s)$ the TLA-coefficient function of f.
  - Overall, in fact, the generalised Césaro framework rather than classical convergence is the correct framework for all this work on TS-to-the-left methods, the definition of $\overset{\vee}{f}(s)$ and integration.
  - As an example, $f(x) = \ln(\Gamma(x+1)) - \frac{1}{12}\frac{1}{x+1}$ is not classically integrable (in light of Stirling's theorem) but is integrable on $[0, \infty)$ in a generalised Césaro sense; the TS-to-the-left methodology continues to work, however, and successfully gives the generalised Césaro value of $\int_0^\infty \left\{\ln(\Gamma(x+1)) - \frac{1}{12}\frac{1}{x+1}\right\} dx$ as $-\overset{\vee}{f}'(-1) = \ln(A)$ where $A \approx 1.282427129$ is the Glaisher-Kinkelin constant.

- We can calculate Mellin transforms using TS-to-the-left methodology together with bootstrapping (to go from $s = \frac{p}{q} \in \mathbb{Q}$ to arbitrary $s \in \mathbb{C}$). In fact this gives us the desired, correct *canonical* form for $\overset{\vee}{f}(s)$ in terms of $\mathcal{M}[f](s)$ as

$$\overset{\vee}{f}(s) = -\mathcal{M}[f](-s) \cdot \frac{\sin(2\pi s)}{2\pi} \quad . \tag{34}$$

  With this canonical choice of $\overset{\vee}{f}(s)$ from among the possible gauge-equivalent options, all the previous claims are readily proved.

- Immediately it becomes clear that the generalised Césaro framework, rather than classical convergence, is also in fact the right framework for treatment and interpretation of all Mellin transform calculations.

- **Final point:** Ramanujan's master theorem (RMT) says that, under appropriate conditions, if $f(x) = \sum_{n=0}^\infty \phi(n) \frac{(-x)^n}{n!}$ for some well-behaved function $\phi(s)$, $s \in \mathbb{C}$, then $\mathcal{M}[f](s) = \Gamma(s) \cdot \phi(s)$. Clearly $\overset{\vee}{f}(s)$ is closely related to $\phi(s)$; the explicit equivalence is that $\phi(s) = \frac{1}{\cos(\pi s)} \cdot s! \cdot \overset{\vee}{f}(s)$. Nonetheless, TS-to-the-left methodology is not redundant and $\overset{\vee}{f}(s)$ is the "right" quantity to work with rather than $\phi(s)$ - it has nicer properties; it can be used for a wider array of integrals than just Mellin transforms; it can be combined as part of a "calculus" of TLA-coefficient functions; we will see that it directly connects the power series behaviour of $f(x)$ near both 0 and ∞; and it lives in the much broader domain of generalised Césaro convergence and so has much wider application.

[Highlight 9 - Connecting behaviour near 0 and ∞]: In fact, the canonical TLA-coefficient function of $f(x)$, $\overset{\vee}{f}(s)$, is a fundamentally global quantity which *simultaneously* connects the behaviour of $f(x)$ for x near 0 and as $x \to \infty$.

- If $f(x) = \sum_{m=m_0}^{\infty} f_0(m)x^m$ for x near 0 and $f(x) = \sum_{m=-\infty}^{m_\infty} f_\infty(m)x^m$ as $x \to \infty$ (these may be Taylor or Laurent or asymptotic series with $R = \infty$, $R > 0$ or $R = 0$) and if $f(x)$ is Césaro-integrable on $[0, \infty)$, then $\overset{\vee}{f}(s)$ exists and $\overset{\vee}{f}(m) = f_0(m) - f_\infty(m)$ for all $m \in \mathbb{Z}$.

- This facilitates radical local-to-global inference as long as we are careful about the possibility of intermingling - i.e. of the possibility of a power, x^m, appearing in the power series for f near both 0 and ∞.

  - For example, we can find a simple function with the prescribed power series expansion $\sum_{j=0}^{\infty} B_{j+1}x^j$ (with $R = 0$) near 0, by reading off $f_0(m)$; inferring $\overset{\vee}{f}(s)$; hence deducing $f_\infty(m)$ and the power series for $f(x)$ near ∞; and algebraically simplifying this. This is how we deduced that $f(x) := \frac{-1}{x} + \sum_{j=1}^{\infty} \frac{1}{(jx+1)^2}$ is such a function (which, in earlier papers, we then confirmed using Césaro arrays).

- The pattern connecting $\overset{\vee}{f}(m)$ to $f_0(m)$ and $f_\infty(m)$ continues in the expected way within the generalised Césaro framework. If $f(x)$ has a term $ax^m \ln x$ in its asymptotic behaviour as $x \to 0$ and a term $bx^m \ln x$ in its asymptotic behaviour as $x \to \infty$, then $\overset{\vee}{f}(s)$ has a simple pole at $s = m$ with residue $(a - b)$ (and equivalently $\mathcal{M}[f](s)$ has a pole of order 2 at $s = -m$ of the form $\frac{-(a-b)}{(s+m)^2}$). Each extra power of $\ln x$ corresponds to an increase by 1 in the order of the poles in $\overset{\vee}{f}(s)$ and $\mathcal{M}[f](s)$. If the terms are of the form $ax^\rho(\ln x)^N$ for $\rho \in \mathbb{C} \setminus \mathbb{Z}$, we still get poles, this time of order $N + 1$ in both $\mathcal{M}[f](s)$ and $\overset{\vee}{f}(s)$.

- It is crucial to understand correctly the allocation of $\overset{\vee}{f}(m)$ between f_0 and f_∞. This is because we have a product rule for TLA-coefficient functions which requires knowing this:

  - If $h(x) = f(x) \cdot g(x)$ then

$$\overset{\vee}{h}(m) = \sum_{j=-\infty}^{\infty} f_0(j)g_0(m-j) - \sum_{j=-\infty}^{\infty} f_\infty(j)g_\infty(m-j) \tag{35}$$

and we can extend to arbitrary $s \in \mathbb{C}$ by

$$\overset{\vee}{h}(s) = \sum_{j=-\infty}^{\infty} f_0(j) g_0(s-j) - \sum_{j=-\infty}^{\infty} f_\infty(j) g_\infty(s-j) \quad . \tag{36}$$

  - In this allocation it is crucial to retain and correctly attribute zero-values, not ignore them! For example, for $f(x) = e^{-x}$ *all* of the values $\overset{\vee}{f}(m)$, $m \in \mathbb{Z}$, arise entirely from f_0 (with $f_\infty(m) \equiv 0$) and it is vital to remember this in calculating the TLA-coefficient function of a product of e^{-x} with any other function (e.g. in an integral calculation), especially if the other function has a non-zero power series expansion near ∞ (See examples in Highlight 10).

- When combined with our canonical relation, working via TLA-coefficient functions and using this product rule may allow easier calculation of the Mellin transform of a product than by direct methods using Mellin contour-integral convolution rules (also see examples in Highlight 10).

[Highlight 10 - The generalised Césaro framework is the true natural habitat of both Mellin transforms and TS-to-the-left methodology]: In the generalised Césaro framework, there is a natural inner-product given by

$$\langle f(x) \mid g(x) \rangle := \int_0^\infty f(x) \cdot g\left(\frac{1}{x}\right) \frac{dx}{x} \tag{37}$$

under which the power functions $\{x^\rho\}_{\rho \in \mathbb{C}}$, which are the eigenfunctions of P, are all orthogonal. The natural "Taylor transform" associated to this set of eigenfunctions and this inner product is then given by

$$\mathcal{T}[f](s) := \langle f(x) \mid x^s \rangle = \int_0^\infty f(x) \cdot x^{-s-1}\, dx \tag{38}$$

and so the Mellin transform is really this natural transform in disguise - i.e. $\mathcal{T}[f](s) = \mathcal{M}[f](-s)$.

- This confirms that the generalised Césaro framework is the right natural habitat for the study of Mellin transforms and TS-to-the-left methodology.
  - In particular, the key orthogonality property that $\langle x^s \mid x^{s_0} \rangle = 0$ whenever $s \neq s_0$ is only well-defined in this generalised Césaro framework and is ill-posed under classical convergence.
  - The fact that $\mathcal{M}[f](s)$, or equivalently $\overset{\vee}{f}(s)$, provides a simultaneous connection between behaviour of $f(x)$ near 0 and near ∞ now becomes natural within this framework.
  - We can now readily prove all our TS-to-the-left results to this point, and the basic formulation of Mellin transform theory is enormously simplified.

- Many of the technical restrictions on domains and invertibility which beset the classical theory of Mellin transforms now evaporate like dew in the morning sun! For example, we now have simply that $\mathcal{M}[e^{-x}](s) = \Gamma(s)$ for all $s \in \mathbb{C} \setminus \mathbb{Z}_{<0}$ and $\mathcal{M}^{-1}[\Gamma(s)](x) = e^{-x}$ on the same domain, rather than having all the successively more and more complicated strip-wise amendments which are required classically.

- All of the core properties of $\mathcal{M}$ continue to hold and we now see more deeply why the Mellin transform relations for dilation and scaling, and for $\left(x \circ \frac{d}{dx}\right)^n f(x)$ and $\left(\frac{d}{dx} \circ x\right)^n f(x)$ are so simple; while the relation for $\left(\frac{d}{dx}\right)^n f(x)$ is much more intricate. It is because $x \circ \frac{d}{dx}$ and $\frac{d}{dx} \circ x$ commute with the Césaro operator P and generalised Césaro convergence is dilation- and scaling-invariant, whereas $\frac{d}{dx}$ is the generator of translations and generalised Césaro convergence is not translation-invariant.

- TS-to-the-left methodology and TLA-coefficient functions can often now make actual calculation of Mellin transforms easier, and even facilitate such calculation where traditional methods encounter difficulties:

  - For example, the derivation of many Mellin transforms is much easier - e.g. $\mathcal{M}[\sin x](s)$ etc.
  - Even where still readily calculated by traditional methods, the alternative calculation of some Mellin transforms using TS-to-the-left methods and TLA-coefficient functions, and their subsequent expression within a generalised Césaro framework, is both simpler and cleaner - e.g. for $\mathcal{M}[H(x)](s)$ where $H(x) = \frac{1}{2} + \sum_{j=1}^{\infty} e^{-\pi j^2 x^2}$.
  - For some functions, like $f(x) = \frac{e^{-x}}{x+1}$ (or more generally $f(x) = \frac{e^{-\mu x}}{x+\beta}$), calculation of $\mathcal{M}[f](s)$ by traditional methods is very challenging, and attempted calculation by expanding either e^{-x} or $\frac{1}{x+1}$ in a Taylor series still fails to give a convergent expression. However, TS-to-the-left methodology succeeds relatively quickly in giving a convergent expression for $\overset{\vee}{f}(s)$ and hence $\mathcal{M}[f](s)$, on using our product rule for TLA-coefficient functions and recalling how to split a finite sum as a difference of two remainder summations in moving from $s = m \in \mathbb{Z}$ to arbitrary $s \in \mathbb{C}$. This derivation also explains the failure of the classical expansion methods to give convergent expressions - it arises from a failure to take proper care in distinguishing and then re-combining g_0/g_∞ and h_0/h_∞ components in the product giving $f(x)$ (i.e. $f(x) = g(x) \cdot h(x)$ with $g(x) = e^{-x}$ and $h(x) = \frac{1}{x+1}$).
  - In other examples (e.g. $f(x) = \frac{1}{(1+x)(1+x^2)}$), the use of TS-to-the-left methodology succeeds, but requires use of Césaro summation on the s-side as well!

- The extension and resulting simplification we have seen for Mellin transform theory by passing to the generalised Césaro framework should be replicable for other transforms via a suitably-adapted generalised convergence scheme ... .

[Highlight 11 - Seeing to the edge of the universe with a microscope]: The global connectedness of behaviour near 0 and near ∞ achieved by TLA-coefficient functions and Mellin transforms within the generalised Césaro framework permits the possibility of radical local-to-global inference, i.e. seeing to the edge of the universe (as $x \to \infty$) based solely on a microscopic examination of an infinitesimal neighbourhood of 0.

- For example, we can deduce Stirling's formula for the asymptotic behaviour of $f(z) = \ln(\Gamma(z+1))$ as $z \to \infty$, namely that

  $$\ln(\Gamma(z+1)) \approx (z + \frac{1}{2}) \ln z - z + \frac{1}{2}\ln(2\pi) + \frac{1}{12}\frac{1}{z} - \frac{1}{360}\frac{1}{z^3} + \frac{1}{1260}\frac{1}{z^5} - \dots \tag{39}$$

  almost trivially just from a consideration of the Taylor series for $\ln(\Gamma(z+1))$ near 0.

- This extremely quick derivation of Stirling's formula (in an argument condensable to a couple of lines) is not possible at any level outside the generalised Césaro framework for Mellin transforms and TLA-coefficient functions, and the TS-to-the-left methodology we have developed for them within it:
  - In the classical convergence world the Mellin transform of $\ln(\Gamma(z+1))$ does not even exist ($\int_0^\infty \ln(\Gamma(z+1)) \cdot z^{s-1}\, dz$ is not classically integrable for any $s \in \mathbb{C}$!);
  - Our derivation of $\mathcal{M}[\ln(\Gamma(z+1))](s)$ within the generalised Césaro framework is done by reading off $\overset{\vee}{f}(s)$ from the Taylor series for $\ln(\Gamma(z+1))$ near 0 and so relies on our whole TS-to-the-left approach;
  - The local-to-global inference to deduce Stirling's formula relies on the global character of $\overset{\vee}{f}(s)$ (i.e. $\overset{\vee}{f}(m) = f_0(m) - f_\infty(m)$), and hence also of $\mathcal{M}[f](s)$. We have only established this as part of TS-to-the-left methodology within the generalised Césaro framework.

[Highlight 12 - Miscellaneous further applications of TS-to-the-left methodology]:
Application (a) [Mellin transforms of p-sum functions]: For a p-sum function, $\sigma(X) := \sum_{x_n < X} a_n$, its TLA-coefficient function, $\overset{\vee}{\sigma}(s)$, or Mellin transform, $\mathcal{M}[\sigma](s)$, immediately gives asymptotic information about $\sigma(X)$ as $X \to \infty$ based on our TS-to-the-left results. This is because σ is identically zero near 0, so that $\overset{\vee}{\sigma}(m) = -\sigma_\infty(m)$, and in general poles in $\overset{\vee}{\sigma}(s)$ or $\mathcal{M}[\sigma](s)$ represent power or power-log terms in the expansion for $\sigma(X)$ as $X \to \infty$.

- Applied to the case of $\sigma_\nu(X) := \sum_{j=1}^k j^\nu$ we have, by our core generalised Césaro identity 1, that $-\mathcal{M}[\sigma_\nu](-s) = -\frac{\zeta(s-\nu)}{s}$, which has a pole at $s = \nu + 1$ with residue $\frac{-1}{\nu+1}$ and a pole at $s = 0$ with residue $-\zeta(-\nu)$.

- It follows that we should have

$$\sigma_\nu(X) \approx \frac{X^{\nu+1}}{\nu+1} + \zeta(-\nu) \quad \text{as } X \to \infty \tag{40}$$

  and again this is true by our core generalised Césaro identity, but only up to a residual function $R_{-\nu}(X)$ which converges to 0 in a strong Césaro sense by some pure power, P^N:

  - i.e. we do have exactly that $\sigma_\nu(X) = \frac{X^{\nu+1}}{\nu+1} + \zeta(-\nu) - R_{-\nu}(X)$ and $R_{-\nu}(X)$ converges to 0 in a strong generalised Césaro sense as $X \to \infty$.

- This is the sense in which we can draw our conclusions about asymptotic behaviour of p-sum functions from calculations of $\overset{\vee}{\sigma}(s)$ or $\mathcal{M}[\sigma](s)$ within our integrated TS-to-the-left framework under generalised Césaro convergence.

- Applied to a famous p-sum function from the theory of ζ such as $\psi(X) := \sum_{p^n<X} \ln p$ we are able to deduce almost immediately that

$$\psi(X) = X - \sum_{\rho \in NT} \frac{X^\rho}{\rho} + \sum_{m=1}^{\infty} \frac{X^{-2m}}{2m} - \frac{\zeta'(0)}{\zeta(0)} + R_\psi(X) \tag{41}$$

  where $R_\psi(X) \overset{C}{\to} 0$ strongly as $X \to \infty$.

- This is almost von Mangoldt's famous formula, except that we have the residual strongly-Césaro-decaying extra term $R_\psi(X)$, while von Mangoldt's exact formula omits any such term but requires pairing of non-trivial roots above and below the real axis in its interpretation, and even then is only conditionally convergent classically.

  - Connecting how these two versions relate to each other - in particular, how the term $R_\psi(X)$ relates to the pairing of roots and the ordering of their sum - would be interesting;
  - Within the generalised Césaro framework, taking equation 41, expanding X^ρ as $\sum_{n=0}^\infty \frac{\rho^n}{n!}(\ln X)^n$ and using Césaro arrays to reverse the order of summation over n and over $\rho \in NT$ is also very much worth pursuing. It leads towards an interesting formula for $\psi(X)$ modulo RH - one which reflects the critical importance of the geometric location of summands in generalised Césaro summation (some stumbling blocks, however, remain).

- Other examples of Césaro Mellin transforms of p-sum functions are also interesting - e.g. for $\sigma_\mu(X) := \sum_{j<X} \mu(j)$; and it would be interesting in general to consider the Mellin transform of $\sigma_a(X) := \sum_{j<X} a_j$ where $\{a_j\}$ is a strongly-periodic sequence of the sort considered in the second paper on Césaro arrays, since sums of the form $\sum_{j=1}^{\infty} a_j \cdot j^s$, which were of great interest in that paper, arise naturally in doing so.

Application (b) [New integral results using TS-to-the-left methods]: Using TLA-coefficient functions and a full range of TS-to-the-left methods (e.g. use of remainder sums to move from finite sums in a discrete variable to a function of arbitrary $s \in \mathbb{C}$; dealing with cases where there is s-dependence in both summand function and summand-location; extending the TLA-coefficient function product rule to products of more than two functions; etc) we are able to derive formulae for a number of definite integrals which we do not otherwise know how to compute by traditional means, and where we do not believe current formulae exist in the literature:

- For example, we can derive a formula for $\int_0^\infty \frac{e^{-x^n}}{1+x^2}\, dx$.
  - This formula is rapidly convergent and elegant for small n.
- Likewise,we can derive a formula in general for $\int_0^\infty e^{-p(x)}\, dx$ for any polynomial of any degree with positive leading coefficient:
  - For $deg(p(x)) \geq 3$ we are unable to find such a formula in the existing literature (e.g. [1]), so we believe this is a new result;
  - The general formula, though intricate, has a clear form which is easily discerned by looking at the cases of $deg(p(x)) = 3$ and $deg(p(x)) = 4$, both of which we give explicitly.
- As noted earlier, in some cases (e.g. $\int_0^\infty e^{-\frac{1}{1+x^k}}\, dx$), what is otherwise a challenging integral by traditional means can be reduced to a very short, simple derivation using only elementary combinatorics and algebra, by combining TS-to-the-left methods with the use of formal symbols and formal function elements (in this case the formal function element, f, for the factorial function, given by $f^s := s!$).

5 Acknowledgements

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