Root Identities I

Richard Stone

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Abstract

It is a truth universally acknowledged that an existing set of identities, already elegantly applicable to the class of polynomials, must be in want of generalisation. We first extend the familiar root identities for polynomials to general functions of a complex variable. We then further extend this set of identities (one for each $\mu \in \mathbb{Z}_{\geq 1}$) to a wider family of "generalised root identities" parametrised by arbitrary $\mu \in \mathbb{C}$, including in particular $\mu \in \mathbb{Z}_{\leq 0}$. These identities link to and generalise Hadamard's theorem, and we consider their meaning in three key examples - $\cos(\frac{\pi z}{2})$, $\Gamma(z+1)$ and $\zeta(s)$ - as well as demonstrating their continued applicability to polynomials. We validate them fully for the cases of $\cos(\frac{\pi z}{2})$ and $\Gamma(z+1)$. In so doing we develop machinery for making sense of the "derivative side" of these identities. This complements the generalised geometric Césaro machinery we have developed previously and which we use for making sense of the corresponding "root side". In all cases, these generalised root identities provide interesting and non-trivial mathematical results, and the cases of $\mu \in \mathbb{Z}_{\leq 0}$ in particular give insight into the asymptotic behaviour of the function in question and the geometric distribution of its roots. We take up the most interesting case of $\zeta(s)$ in subsequent papers.

1 Background and map for this set of papers on root identities

This is the first in a set of four papers developing the notion of generalised root identities and their application.

In [I] - [III] we introduced generalised geometric Césaro convergence as a general method for performing constructive analytic continuation of complex functions outside their domains of natural definition.

In [IV] - [VI] we then applied this generalised Césaro framework in a systematic way to develop the methodology of 2-d Césaro arrays.

In this set of papers we explore a second area of application of generalised geometric Césaro methods - in what we call *generalised* root identities. These generalise the familiar root identities for polynomials which, for a degree n polynomial $p(z)$, are a collection of n identities relating the symmetric polynomials in the roots of p to simple ratios of its coefficients.

For smooth non-polynomial functions these *generalised* root identities equate an expression involving derivatives of the log of the function (this is the derivative side of the identity) with a certain sum over its roots (called the root side and understood more generally to include also poles and even branch points). Initially this family of generalised identities is parametrised by $\mu \in \mathbb{Z}_{\geq 1}$ but we then further generalise to arbitrary $\mu \in \mathbb{C}$, with the cases $\mu \in \mathbb{Z}_{\leq 0}$ being of particular interest. Making sense of the derivative side of these generalised root identities requires us to invoke some machinery - fractional calculus, or Fourier transform and distribution theory - to give it meaning; while on the root side we need to invoke generalised geometric Césaro convergence in order to make sense of the series involved, in regions of the μ-plane where they are classically divergent.

In the first paper in the set, we derive heuristically what form these identities should take and develop the machinery required to make sense of them. The following three papers then go together and explore in depth the application of these identities to the Riemann zeta function - culminating in a new family of identities for ζ with interesting implications.

As always, our focus initially is on following computational roads and tracks which demonstrate the utility and value of generalised root identities - rather than allowing our horses to stray too early into muddy fields overgrown with thorny questions of precise conditions and abstract rigour - but all claims are ultimately justified rigorously.

Paper I: In this first paper we thus concentrate principally on developing fully three example cases - the generalised root identities for the functions $\cos(\frac{\pi z}{2})$, $\Gamma(z+1)$ and the case of polynomials where we began - in addition to providing a preliminary consideration of the case of $\zeta(s)$.

On the root side of our identities, the necessary machinery of geometric generalised Césaro remainder convergence has been laid out fully in [I] - [III]. On the derivative side, we use our examples to organically develop the required machinery. Note, however, that it is not necessary to be an expert in Fourier theory, distribution theory or fractional calculus to be able to read this paper and subsequent papers in this series. Rather, we emphasize a small number of properties, and a handful of key results regarding Fourier transforms and distribution theory, which suffice to unlock the derivative side in our examples and more generally.

Our three examples reveal that generalised root identities represent a natural extension of Hadamard's theorem giving a canonical factorisation for entire functions of finite order. They are an extension both to a broader class of functions, in light of the relaxation of conditions which becomes permissible in moving on the root side from a classical to a generalised Césaro convergence framework; and to a family of identities parametrised not just by $\mu \in \mathbb{Z}_{\geq 1}$ but now by arbitrary $\mu \in \mathbb{C}$, including especially the cases of $\mu \in \mathbb{Z}_{\leq 0}$.

These latter cases of $\mu \in \mathbb{Z}_{\leq 0}$ are of particularly interest because on the root side they entail a formal, renormalised "count" of roots (when $\mu = 0$) and then sums of higher powers of (shifted) roots (when $\mu = -1, -2, \ldots$). Such divergent sums are naturally non-local and thus give information about the behaviour of the underlying function $f(z)$ as $z \to \infty$ and about the asymptotic distribution of its roots.

Since, moreover, our generalised Césaro framework is also critically dependent on the geometric *location* of summands - in this case of roots of f - the generalised root identities at $\mu \in \mathbb{Z}_{\leq 0}$ provide a natural tool for investigating the geometric location of the roots of a function, and one which gives as much weight to roots approaching ∞ as it does to any finite root.

It is this observation (which we shall explain more fully later in this paper) which animates the next papers in this set. For while we prove in this paper that the generalised root identities are satisfied by polynomials, by $\cos(\frac{\pi z}{2})$ and by $\Gamma(z+1)$ (and demonstrate that, in all cases, this fact contains highly non-trivial mathematical information), we defer any corresponding proof for ζ beyond the cases of $\mu \in \mathbb{Z}_{\geq 1}$. But of course the most famous question regarding ζ - namely the Riemann hypothesis (RH) - postulates a claim about the geometric location of its roots, specifically that all its non-trivial roots lie on the critical line $Re(s) = \frac{1}{2}$. As such ζ is precisely the sort of function which it would be natural to investigate using our generalised root identities, especially in the cases of $\mu \in \mathbb{Z}_{\leq 0}$.

Papers IIA - IIC: Based on this perspective, the final three papers in this series thus focus exclusively on the generalised root identities for ζ, culminating in the derivation of a family of new results regarding ζ and the RH. Specifically we derive a family of new integral identities for the argument of the zeta function, $S(T)$, or equivalently a new result regarding its Mellin transform, conditional on RH.

In the first of these three papers we continue with our practical and computational focus. After deriving the analytic form of the derivative side of our identities for ζ and giving convincing *numerical* evidence that they are satisfied, we perform explicit Césaro calculations of the root side for $\mu = 0$ and $\mu = -1$ unconditionally; and then also for $\mu = -2$ under an assumption of RH. These calculations validate the correctness of the generalised root identities for these three μ-values (at least modulo RH in the last instance).

They also illustrate explicitly the *criticality* of the geometric location of the roots of ζ for the root-side calculations when $\mu \in \mathbb{Z}_{\leq 0}$, and give an insight into how similar calculations for more negative integer μ-values would proceed. In particular, they show how the non-trivial roots above the real axis and their mirror-counterparts below it lead to cancelling poles in μ, and also how they combine, as required, to perfectly balance the contributions from the pole of ζ and from its trivial roots. The case of $\mu = -2$ also gives the first of our explicit new integral results for ζ and its argument function $S(T)$.

In our second paper we then pause and turn back from a calculational focus to *prove* rigorously that, modulo a small obstruction at $\mu = 1$, ζ does indeed satisfy the generalised root identities at all $\mu \in \mathbb{C}$. We do this via an argument using fractional calculus and the paper is consequently both self-contained and relatively short.

In the last of the three papers on generalised root identities for ζ, we then turn back to a computational focus and systematically extend the earlier calculations for $\mu = 0, -1$ and -2 to evaluate the root side of our identities for $\mu = -3, -4, \ldots$, at least conditionally on an assumption of RH. By invoking the known value of the derivative side of the root identities at these μ-values we are able to deduce our promised new result regarding ζ, namely a family of integral identities for $S(T)$, or equivalently a general result regarding the Mellin transform of $S(T)$, conditional on RH.

We conclude by deducing some further implications of this result - in particular, some general results for the asymptotics of integrals against $dS(t)$ obtained by connecting to Césaro arrays; and we discuss its relevance regarding the truth or falsity of the RH itself, including discussion of how the computations in these three papers might vary if we did not assume RH at the outset.

This then is the road map for our suite of papers on generalised root identities. We now outline in detail the particular country lanes and byways we will explore in this first instalment.

2 Introduction to this paper

In section 3.1 we heuristically derive the form of our generalised root identities for an arbitrary smooth function $f(z)$. Initially these give a family of such identities for arbitrary reference point z_0 and $\mu \in \mathbb{Z}_{\geq 1}$. These equate a quantity, $d_f(z_0, \mu)$, defined in terms of the derivatives of the log of the function (the "derivative side") with a quantity, $r_f(z_0, \mu)$, defined as a sum over the (shifted) roots of the function (the "root side"). The root side in fact includes equally contributions from roots (with multiplicity $M \in \mathbb{Z}_{\geq 1}$), from poles (with $M \in \mathbb{Z}_{<0}$) and even from branch points (with e.g. $M = \frac{1}{2}$). A natural equivalence is then established between a function satisfying the root identities at a single z_0 for all $\mu \in \mathbb{Z}_{\geq 1}$, and satisfying them at arbitrary $z \in \mathbb{C}$ (at least in an open neighbourhood of z_0) for $\mu = 1$ alone.

Unfortunately it is readily seen that most non-polynomial functions do not satisfy these generalised root identities! However, where the discrepancy is sufficiently well-behaved (as a function of z_0), we show in subsection 3.1.1 that it is possible to find an "equivalent" function with the same root-set which does satisfy these identities. We define this equivalence class and show how we can construct this unique representative ("equivalent function") within any class in an engineering fashion, by successively removing obstructions to the generalised root identities for $\mu = 1, 2, 3, \ldots$ at a given fixed z_0.

In section 3.2, however, we verify that in practice many natural and important functions already satisfy these identities with no, or minimal, adjustment! We illustrate with three examples which will become primary focuses of attention in this and succeeding articles, showing that the functions $\cos(\frac{\pi z}{2})$, $\Gamma(z+1)$ and $\zeta(s)$ all satisfy the generalised root identities for general $z_0 \in \mathbb{C}$ and all $\mu \in \mathbb{Z}_{\geq 1}$. In the latter two cases there are technical issues to resolve when $\mu = 1$ - a renormalisation on the root side for $\Gamma(z+1)$ and the removal of an obstruction for $\zeta(s)$ - but these are confined to this single instance and the identities hold without caveat for $\mu = 2, 3, 4, \ldots$.

In the case of $\zeta(s)$, this obstruction removal for $\mu = 1$ links us directly to the Hadamard formula for $\xi(s)$ and in general we see in all three of our examples that the generalised root identity for $\mu = 1$ (and hence also for $\mu = 2, 3, \ldots$ by differentiation) is closely connected to Hadamard's general factorisation theorem for entire functions of finite order.

In section 4 we turn to further generalising our root identities by allowing μ to be not just a positive integer, but $\mu \in \mathbb{C}$ arbitrary, with the cases of $\mu \in \mathbb{Z}_{\leq 0}$ being of particular interest.

In section 4.1 we begin by discussing how to even make sense of the identities in the case of arbitrary complex μ.

On the root side a generalised convergence scheme - always Césaro in these papers - is required to make sense of what are now often divergent sums, and we find that it is crucial that the summands, $\frac{M_i}{(z_0 - r_i)^\mu}$, in this Césaro context are added in *geometrically* at the shifted points $z_0 - r_i$.

On the derivative side the notion of $\left(\frac{d}{dz}\right)^\mu$ now becomes problematic and we begin by noting two basic properties it should have. These alone suffice to successfully accomplish the extension of our generalised root identities to arbitrary complex μ in the case of our first example function $f(z) = \cos(\frac{\pi z}{2})$. We show that f satisfies the generalised root identities for all z_0 and all $\mu \in \mathbb{C}$ and that the content of these generalised root identities in this case is in fact equivalent to the functional equation for ζ.

However, these properties alone do not suffice to make sense of the derivative side of our generalised root identities for our other two examples. We thus turn in section 4.2 to defining $\left(\frac{d}{dz}\right)^\mu$ more generally (but in a way consistent with these properties). We use Fourier theory, in a manner familiar from the definition of pseudo-differential operators. We state the key results regarding Fourier transforms which we shall rely on for arbitrary μ, and we also note the key result from distribution theory which we shall invoke when considering the important special cases of $\mu \in \mathbb{Z}_{\leq 0}$. This distributional result concerns the interpretation of expressions of the form $\frac{\xi_+^\mu}{\Gamma(\mu+1)}$ when $\mu \in \mathbb{Z}_{\leq 0}$ and makes calculation for such μ tractable. It is critical in explaining how $d_f(z_0, \mu)$ can often be non-trivial for $\mu \in \mathbb{Z}_{\leq 0}$ even though the factor $\Gamma(\mu + 1)$ on the denominator on the derivative side becomes singular.

In section 4.3 we then illustrate the effectiveness of this interpretation of the derivative side. We begin by applying it in subsection 4.3.1 to the class of functions we started with - polynomials - and verifying the non-trivial fact that these still satisfy the generalised root identities even for $\mu \in \mathbb{C} \setminus \mathbb{Z}_{\geq 1}$.

In subsection 4.3.2 we then turn to our second example function $\Gamma(z + 1)$. We show that it too satisfies the generalised root identities for all $\mu \in \mathbb{C}$ (modulo the previously-discussed renormalisation required at $\mu = 1$). We demonstrate explicitly how the non-trivial outcomes at $\mu \in \mathbb{Z}_{\leq 0}$ arise in this Fourier/distributional framework and we make a number of comments arising from these calculations about this framework and our generalised root identities in general. These include observations that even on the derivative side, distributional results can often be more easily calculated by naive termwise Césaro-evaluation of the integrals involved; and that the root identities at $\mu \in \mathbb{Z}_{\leq 0}$ naturally yield much more information than the corresponding cases of $\mu \in \mathbb{Z}_{\geq 1}$, both about the behaviour of a function $f(z)$ as $z \to \infty$ and about the asymptotic and geometric distribution of its roots.

As a direct illustration of this last point, we conclude in section 4.4 by showing how we can use the generalised root identities for $\Gamma(z+1)$ at $\mu \in \mathbb{Z}_{\leq 0}$ to engineer Stirling's famous formula giving the asymptotic behaviour of Γ as $z \to \infty$.

At this point, of course, we begin to catch glimpses of the sweeping grounds, gravel paths and stately architecture of $\zeta(s)$, and to ask whether the generalised root identities apply within her environs and, if so, whether they lead to any new vistas or hitherto unexplored rooms. However, out of natural delicacy of manners and a concern for the nerves of the reader, we defer such investigations to the next papers.

3 Generalised root identities

3.1 A first generalisation of root identities

For a polynomial $p(z) = \sum_{j=0}^{n} a_j z^j$ it is trivial that the roots are related to the coefficients by $\sum_{\{roots\, r_i\}} r_i = -\frac{a_{n-1}}{a_n}$, with similar identities for $\sum r_i^m$, $m = 2, 3, \ldots, n$.[1]

These identities can be recast in a way potentially applicable to more general functions - for example entire functions with Taylor series $\sum_{n=0}^{\infty} a_n z^n$ - as relations involving the reciprocals of the roots, as follows.

Suppose $f(z) = N \cdot \prod (z - r_i)^{M_i}$; then $\ln(f(z)) = \ln N + \sum M_i \ln(z - r_i)$ and in general we should have the following "generalised root identity":

$$-\frac{1}{\Gamma(\mu)} \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{\mu} (\ln(f(z)) \Big|_{z=z_0} = (-1)^{\mu} \sum_{\{roots\, r_i\}} \frac{M_i}{(z_0 - r_i)^{\mu}} \tag{1}$$

for any $\mu \in \mathbb{Z}_{\geq 1}$ and $z_0 \in \mathbb{C}$. In this formula note that nothing requires $M_i \in \mathbb{Z}_{>0}$. The multiplicities, M_i, may be negative integers (r_i a pole) or indeed arbitrary complex numbers (r_i a branch point). We shall continue to use the term roots, or sometimes generalised roots, to cover all these possibilities.

Note also that while equation 1 gives a family of identities for $\mu = 1, 2, 3, \ldots$ at any given z_0, this is of course equivalent to knowing the single identity for $\mu = 1$ at general z in a neighbourhood of z_0. For if equation 1 is satisfied for all $\mu \in \mathbb{Z}_{\geq 1}$ at z_0, then for $\mu = 1$, at any $z_0 + h$ within the radius of convergence of

[1] or equivalently for the symmetric polynomials in the roots

the Taylor series of f around z_0, we have in equation 1 that

$$\begin{aligned}
LHS &= -\frac{\mathrm{d}}{\mathrm{d}z}(\ln(f(z)))\mid_{z=z_0+h} = -\frac{\mathrm{d}}{\mathrm{d}z}(\ln(\tilde{f}(z)))\mid_{z=z_0} \text{ where } \tilde{f}(z) = f(z+h) \\
&= -\frac{\mathrm{d}}{\mathrm{d}z}\left(\mathrm{e}^{h\frac{\mathrm{d}}{\mathrm{d}z}}(\ln(f(z)))\right)\mid_{z=z_0} \\
&= -\sum_{j=0}^{\infty}\frac{h^j}{\Gamma(j+1)}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{j+1}(\ln(f(z)))|_{z=z_0} \\
&= \sum_{j=0}^{\infty}(-1)^{j+1}h^j\sum_{\{roots\, r_i\}}\frac{M_i}{(z_0-r_i)^{j+1}} \\
&= -\sum_{\{roots\, r_i\}}\frac{1}{(z_0-r_i)}\cdot M_i\cdot\frac{1}{\left(1-\frac{h}{(r_i-z_0)}\right)} \\
&= -\sum_{\{roots\, r_i\}}\frac{M_i}{((z_0+h)-r_i)} = RHS
\end{aligned}$$

and vice-versa.

3.1.1 Range of validity - equivalence classes and obstructions

Unfortunately, of course, while both sides of equation 1 now make sense for many more general functions, this "generalised root identity" is generically untrue for non-polynomials! For example, for $f(z) = e^z$, f has no roots so the RHS is always zero, while for $\mu = 1$ the LHS is identically 1 for any z_0.

However, suppose f fails equation 1 at $\mu = 1$ with error function

$$g(z) = -\frac{f'(z)}{f(z)} + \sum_{\{roots\, r_i\, of\, f\}}\frac{M_i}{(z-r_i)} \tag{2}$$

an entire function. Then if we set $h(z) = e^{G(z)}f(z)$ where $G'(z) = g(z)$ it follows from the fact that the root-sets of f and h are identical that in equation 1 for $h(z)$ we have

$$\begin{aligned}
LHS &= -\frac{\mathrm{d}}{\mathrm{d}z}(\ln(h(z)) = -\frac{\mathrm{d}}{\mathrm{d}z}(G(z)+\ln(f(z))) \\
&= -g(z) - \frac{f'(z)}{f(z)} = -\sum_{\{roots\, r_i\, of\, h\}}\frac{M_i}{(z-r_i)} = RHS \quad .
\end{aligned}$$

Thus, even though f fails the generalised root identity 1, there is a unique (up to an overall scalar), nowhere-zero entire function $e^{G(z)}$ whose product with $f(z)$ yields a function with the same root-set satisfying the identity.

This is best expressed in terms of equivalence classes as follows:

Definition: *We set two functions equivalent, $f \sim h$, if and only if there exists an entire, nowhere-zero function, $k(z)$, with $k(0) = 1$ such that $h(z) = k(z)f(z)$.*

What we have shown is that within any equivalence class there is a unique representative which satisfies the generalised root identities equation 1.

This can be viewed in a different way by thinking of z_0 fixed and taking $\mu = 1, 2, 3, \ldots$ successively. Taking $z_0 = 0$ for example, suppose the LHS and RHS of equation 1 differ by a_1 for $\mu = 1$. Then we can remove this "obstruction" by multiplying f by $e^{a_1 z}$ since this leaves the root side undisturbed and contributes the required a_1 to the derivative side.

Similarly, if the obstruction for the $\mu = n$ identity at $z_0 = 0$ is a_n then multiplying f by $\exp(\frac{a_n z^n}{n!})$ again leaves the root side of equation 1 unchanged but contributes precisely the required correction of a_n to the derivative side in the $\mu = n$ identity; moreover, this contributes nothing to any of the other identities with $\mu \in \mathbb{Z}_{\neq n}$ and thus leaves these identities undisturbed.

In this way, working through $\mu = 1, 2, 3, \ldots$ we can successively correct each obstruction and produce a new function $h(z) = \exp(a_1 z + \frac{a_2 z^2}{2!} + \frac{a_3 z^3}{3!} + \ldots) \cdot f(z)$ which has the same generalised root-set as f and does satisfy the root identities equation 1 for all $\mu \in \mathbb{Z}_{\geq 1}$.

Note that if the error function, $g(z)$, obstructing the $\mu = 1$ identity for f is not entire, then the situation becomes more delicate.

3.2 Examples

In reality the root identities 1 hold with little or no need for adjustment for many well-known functions. In this section we briefly consider three examples.

Example 1 [$f(z) = \cos(\frac{\pi z}{2})$]: Here the roots of f are all simple roots of multiplicity 1 at the points $\pm(2k-1)$, $k \in \mathbb{Z}_{>0}$, and so for $\mu = 1$ at arbitrary z_0 in equation 1, we have

$$\begin{aligned} RHS &= -\sum_{k=1}^{\infty} \frac{2z_0}{z_0^2 - (2k-1)^2}\,, \quad \text{while} \\ LHS &= \frac{\pi}{2} \tan(\frac{\pi z_0}{2}) \end{aligned}$$

These are equal by a well-known identity, and so $f(z) = \cos(\frac{\pi z}{2})$ satisfies the generalised root identities 1 for arbitrary z_0 and all $\mu \in \mathbb{Z}_{\geq 1}$.

Since the roots of f are closely related to the positive integers, the RHS of 1 leads easily to values of $\zeta(n)$, $n \in \mathbb{Z}_{>0}$. In particular, for $\mu = 2$ the root identity at $z_0 = 0$ immediately captures Euler's famous formula that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. This is an example of an "inverse" use of the generalised root identities, where a function of interest, ζ, arises naturally from the root side of the identities for a known function, $f(z) = \cos(\frac{\pi z}{2})$, and information about ζ is deduced by considering the derivative side of the identities for f. We shall return to this example, along these lines, in the next section.

Example 2 **[$\Gamma(z+1)$]:** The roots of $\Gamma(z+1)$ are all simple poles ($M_i = -1$) at $z = -1, -2, -3, \ldots$. Thus for $\mu = 1$ and arbitrary z_0 in 1, we have

$$RHS = \sum_{n=1}^{\infty} \frac{1}{z_0 + n} \quad .$$

This is divergent for all z_0 and is not even Césaro summable because the partial sums involve the generalised Césaro eigenfunction with eigenvalue 1, $\ln z$. However, since this Césaro-obstruction is uniform, it can be "renormalised" away uniformly as

$$RHS = \sum_{n=1}^{\infty} \left(\frac{1}{z_0 + n} - \frac{1}{n} \right) + \gamma \quad . \tag{3}$$

Here $\gamma \approx 0.577$ is the Euler-Mascheroni constant and arises because the difference between the $\ln z$ divergence we are removing and the partial sums of $\sum \frac{1}{n}$ approaches γ in the limit ($\lim_{N \to \infty} (\sum_{n=1}^{N} \frac{1}{n} - \ln N) = \gamma$).

But equation 3 is in turn a well-known expression for $-\frac{\Gamma'(z+1)}{\Gamma(z+1)}|_{z=z_0}$ and thus we see that, after renormalisation of the Césaro log-divergence, $\Gamma(z+1)$ does satisfy the generalised root identity 1 for $\mu = 1$ and arbitrary z_0 (note that the expression in 3 retains the required simple poles at $\mathbb{Z}_{<0}$).

The need to perform this renormalisation adjustment arises only for the case of $\mu = 1$. For $\mu = 2$ the RHS of the generalised root identity 1 is $-\sum_{n=1}^{\infty} \frac{1}{(z_0+n)^2}$, which is classically convergent for all z_0 and clearly equals the the LHS, namely $-\frac{\mathrm{d}^2}{\mathrm{d}z^2}(\ln(\Gamma(z+1)))|_{z=z_0}$, on differentiating the expression for $-\frac{\Gamma'(z+1)}{\Gamma(z+1)}|_{z=z_0}$ in equation 3; likewise for $\mu = 3, 4, \ldots$.

Thus overall $\Gamma(z+1)$ satisfies the generalised root identities 1 for arbitrary z_0 and all $\mu \in \mathbb{Z}_{\geq 1}$, albeit after requiring renormalisation when $\mu = 1$ to uniformly remove the Césaro non-amenable log-divergence in this case. It is easy to see that the same holds true for the case of general $\Gamma(az+b)$.

Example 3 **[$\zeta(s)$]:** The root set of $\zeta(s)$ consists of the trivial zeros at $s = -2, -4, \ldots$ (with $M_i = 1$), the simple pole at $s = 1$ (with $M_i = -1$) and the famous non-trivial zeros in the critical strip $0 < Re(s) < 1$ (for which $M_i > 0$ are unknown in general). We let T denote the set of trivial roots and NT the set of non-trivial roots, which we will also denote by ρ_i rather than r_i. If the Riemann hypothesis is true, then the ρ_i occur in conjugate pairs solely on the critical line $s = \frac{1}{2}$; if not then some occur in quadruples via reflection in the real axis and critical line: ρ_i, $\bar{\rho}_i$, $1 - \rho_i$, and $1 - \bar{\rho}_i$.

Of course, the roots in NT are not known exactly and thus tackling the root side of the generalised root identities 1 directly is difficult. However, working empirically first, we can take a list of, say, the first 100,000 non-trivial zeros and use them to test experimentally whether ζ seems to satisfy equation 1 for $\mu = 1$. In doing this, however, a renormalisation analogous to the last example will have to be carried out to handle the trivial zeros, T. Considering the case

$s_0 = 0$ initially, this is equivalent to setting $\sum_T \frac{1}{(0-r_i)} = \frac{\gamma}{2}$ (the $\frac{1}{2}$ factor arising naturally since T only covers the negative even integers).

Thus, taking initially only the truncated non-trivial root-set $\widetilde{NT}$ consisting of the first 100,000 non-trivial zeros[2] (see [7]), we find that the root side of equation 1 for $\mu = 1$ at $s_0 = 0$ becomes

$$RHS \approx -\{\frac{\gamma}{2} + 1 + \sum_{\widetilde{NT}} \frac{1}{(0-\rho_i)}\} \approx -1.2655342$$

(on performing the last computation numerically to obtain $\sum_{\widetilde{NT}} \frac{1}{\rho_i} \approx 0.0230737$).

By contrast the derivative side in equation 1 gives $-\frac{\zeta\prime(0)}{\zeta(0)} = -\ln(2\pi) \approx -1.8378771$. At once we therefore see that ζ does not directly satisfy the generalised root identities for $\mu = 1$ at $s_0 = 0$, with the obstruction in this instance being, numerically, $0.5723429 \approx 0.5723649 = \frac{1}{2}\ln\pi$.[3]

Next consider $s_0 = \frac{1}{2}$. Here, by the symmetry outlined above, $\sum_{NT} \frac{1}{(\frac{1}{2}-\rho_i)} = 0$ and so $RHS = -\{\sum_T \frac{1}{(\frac{1}{2}-r_i)} - \frac{1}{(\frac{1}{2}-1)}\}$. In this case the calculation of the renormalised value of the first sum is best done by noting that the trivial zeros occur at the same locations as the poles of $\Gamma(\frac{s}{2}+1)$ and thus, up to an overall factor of -1, this sum can be evaluated using the generalised root identity for $\Gamma(\frac{s}{2}+1)$ at $s_0 = \frac{1}{2}$, namely as $\frac{1}{2}\frac{\Gamma'(\frac{5}{4})}{\Gamma(\frac{5}{4})} = -\frac{1}{2}\{\sum_{n=1}^{\infty}(\frac{1}{(n+\frac{1}{4})} - \frac{1}{n}) + \gamma\}$. We thus obtain $RHS = -\frac{\gamma}{2} - \frac{\pi}{4} - \frac{3}{2}\ln 2$, while $LHS = -\frac{\zeta'(\frac{1}{2})}{\zeta(\frac{1}{2})} = -\frac{\gamma}{2} - \frac{\pi}{4} - \frac{3}{2}\ln 2 - \frac{1}{2}\ln(\pi)$, so that again we obtain the same value of $\frac{1}{2}\ln(\pi)$ as an obstruction.

This suggests, along the lines outlined before, that while ζ does not directly satisfy the generalised root identities 1 for $\mu = 1$, the function $\pi^{-\frac{s}{2}}\zeta(s)$ may do so.

This conjecture can be expressed in a different way. Recall that the functional equation for ζ can be re-expressed as simply

$$\xi(s) = \xi(1-s) \quad \text{where} \quad \xi(s) := (1-s)\Gamma(\frac{s}{2}+1)\pi^{-\frac{s}{2}}\zeta(s) \tag{4}$$

Since the generalised root identities 1 hold for $(1-s)$ and $\Gamma(\frac{s}{2}+1)$ (as discussed in example 2); and since a product of functions satisfying these identities will also satisfy these identities; so the conjecture that $\pi^{-\frac{s}{2}}\zeta(s)$ satisfies them for $\mu = 1$ and arbitrary s_0 is equivalent to having ξ satisfy 1 for $\mu = 1$ and arbitrary s_0.

In fact this conjecture turns out to be true. In light of the discussion of Γ in the previous example it is equivalent to the following result which can be found on page 35 in [8]:

Theorem 1: *Let NT denote the set of non-trivial zeros of ζ, and NT_+ denote*

[2] actually 200,000 including conjugates

[3] This numerical approximation of the obstruction can be made arbitrarily close to the true value of $\frac{1}{2}\ln\pi$ by increasing the number of NT roots used in the calculation

the subset with imaginary part > 0. Then, with sums understood to include multiplicities, we have

$$-\frac{\zeta'(s)}{\zeta(s)} = \left\{ \begin{array}{c} \frac{1}{s} + \frac{1}{s-1} + \frac{1}{2}\frac{\Gamma'(\frac{s}{2})}{\Gamma(\frac{s}{2})} \\ \\ -(2s-1)\sum_{\rho \in NT_+} \frac{1}{(s-\rho)(s-(1-\rho))} - \frac{1}{2}\ln\pi \end{array} \right\} \tag{5}$$

Thus, overall, both $\pi^{-\frac{s}{2}}\zeta(s)$ and $\xi(s)$ satisfy the generalised root identities 1 for $\mu = 1$ and arbitrary s_0, and hence also for arbitrary $\mu \in \mathbb{Z}_{>1}$. Since the factor $\pi^{-\frac{s}{2}}$ only contributes to the root identities when $\mu = 1$, note that for $\mu \in \mathbb{Z}_{>1}$ the root identities will in fact be satisfied directly by ζ (which can of course be readily spot-checked experimentally).

Comment: Theorem 1 is in fact the expression of the Hadamard product formula for ξ (which is a holomorphic integral function of order 1). We thus see that in this case the generalised root identity 1 for $\mu = 1$ is equivalent to the Hadamard product formula for ξ. Since the identities for $\mu \in \mathbb{Z}_{>1}$ are just derivatives of the $\mu = 1$ identity, they will contain no additional information, but we turn now to considering the case of more general $\mu \in \mathbb{C}$, in particular $\mu \in \mathbb{Z}_{\leq 0}$, in the hope that it will lead to new tools with additional content beyond the Hadamard identity.

4 Further generalisation of the root identities

Our current root identities in equation 1 constitute a generalisation from the case of polynomials. They relate derivatives of $\ln(f(z))$ at arbitrary z_0 to sums of *integer* powers of shifted reciprocals of (generalised) roots of f (for a suitable representative function f within each equivalence class). It is natural, next, to ask whether they can be further generalised by allowing μ to be an arbitrary complex number rather than just a positive integer; that is

$$\frac{-1}{\Gamma(\mu)}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{\mu}(\ln(f(z)))\Big|_{z=z_0} = \mathrm{e}^{i\pi\mu}\sum_{\{roots\, r_i\}}\frac{M_i}{(z_0 - r_i)^{\mu}}, \quad \mu \in \mathbb{C} \tag{6}$$

Equation 6 represents the final, fully general form of our generalised root identities. We call the LHS here the derivative side of these root identities and denote it by $d_f(z_0, \mu)$ (or just $d(z_0, \mu)$ if the context is clear). The RHS is the root side and is denoted by $r_f(z_0, \mu)$ (or just $r(z_0, \mu)$); and equation 6 is interpreted as asserting the identity of these two functions of two complex variables.

4.1 Making sense of the full generalised root identities - first steps

Of course, in attempting the extension given in equation 6 the question immediately arises of how to interpret either side of these identities when μ is no longer a positive integer:

(a)(i) [Root side Césaro treatment]: To overcome the potential divergence of the sum on the root side (e.g. when $Re(\mu)$ becomes negative) the RHS must be interpreted via a generalised convergence scheme, which analytically continues the RHS from its region of convergence in the μ-plane. In all cases in this suite of papers this will be a generalised Césaro scheme.

(a)(ii) [Geometry of roots]: In implementing this Césaro approach, the sum on the root-side will need to be interpreted *geometrically*, in line with the theory developed in [I] - [III]. Specifically, each term $\frac{M_i}{(z_0-r_i)^\mu}$ must be added in at the shifted point $z_0 - r_i$ itself in the complex plane (rather than, for example, always being added in at r_i itself). Thus, as z_0 varies, not only do the summands vary, but their locations move too. We shall often emphasise this by writing $\sum_{\{z_0 - roots\ r_i\}}$ on the RHS in 6.

(b)(i) [Interpretation of $\left(\frac{d}{dz}\right)^\mu$ - first steps]: On the derivative side in 6, the interpretation of $\left(\frac{d}{dz}\right)^\mu$ also now becomes problematic. Two facts which should hold in any such definition, however, are that

$$\left(\frac{d}{dz}\right)^\mu (a^z)\Big|_{z=z_0} = a^{z_0}(\ln a)^\mu \tag{7}$$

and

$$\frac{1}{\Gamma(\mu+1)}\left(\frac{d}{dz}\right)^\mu (z^\rho)\Big|_{z=0} = \begin{cases} 1 \quad , & \rho = \mu \\ 0 & , \quad else \end{cases} \tag{8}$$

We shall formalise the definition of the LHS of 6 in a manner consistent with these criteria below. First, however, we demonstrate the extra power obtained from this extension of the generalised root identities to arbitrary complex μ by showing how the full functional equation for ζ follows very simply from applying them to the function $f(z) = \cos(\frac{\pi z}{2})$ considered in example 1 in section 3.2.

4.1.1 The full root identities for $f(z) = \cos(\frac{\pi z}{2})$ and the functional equation for ζ

In this case, taking $z_0 = 0$ in 6 we have

$$RHS = r_f(0,\mu) = e^{i\pi\mu}(1 + e^{-i\pi\mu})(1 - 2^{-\mu})\zeta(\mu)$$

while for $\mu \neq 0, 1$ we have,on invoking equation 7, that

$$\begin{aligned} LHS = d_f(0,\mu) &= -\frac{1}{\Gamma(\mu)}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^\mu \left(\ln\left(\frac{\mathrm{e}^{\frac{i\pi z}{2}} + \mathrm{e}^{-\frac{i\pi z}{2}}}{2}\right)\right)\Big|_{z=0} \\ &= -\frac{1}{\Gamma(\mu)}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^\mu \left\{-\frac{i\pi z}{2} + \ln(1 + \mathrm{e}^{i\pi z})\right\}\Big|_{z=0} \end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{\Gamma(\mu)}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{\mu}\left\{\mathrm{e}^{i\pi z}-\frac{\mathrm{e}^{2i\pi z}}{2}+\frac{\mathrm{e}^{3i\pi z}}{3}-\ldots\right\}\Big|_{z=0} \\
&= -\frac{1}{\Gamma(\mu)}\left\{(i\pi)^{\mu}-\frac{(2i\pi)^{\mu}}{2}+\frac{(3i\pi)^{\mu}}{3}-\ldots\right\} \\
&= -\frac{\mathrm{e}^{\frac{i\pi\mu}{2}}\pi^{\mu}}{\Gamma(\mu)}\left\{1^{\mu-1}-2^{\mu-1}+3^{\mu-1}-\ldots\right\} \\
&= -\frac{\mathrm{e}^{\frac{i\pi\mu}{2}}\pi^{\mu}}{\Gamma(\mu)}(1-2^{\mu})\zeta(1-\mu) \quad .
\end{aligned}$$

On setting $d_f(0,\mu) = r_f(0,\mu)$, we obtain

$$\zeta(1-\mu) = 2^{1-\mu}\pi^{-\mu}\cos(\frac{\pi\mu}{2})\Gamma(\mu)\zeta(\mu) \tag{9}$$

which is precisely the functional equation for ζ. Thus the functional equation for ζ is equivalent to $\cos(\frac{\pi z}{2})$ satisfying the generalised root identities 6 (after also extending easily to the cases of $\mu = 1$ and $\mu = 0$).

4.2 Making sense of the full generalised root identities - final steps

The properties 7 and 8 which have sufficed to make sense of $d_f(z_0,\mu)$ for $f(z) = \cos(\frac{\pi z}{2})$, do not, however, suffice to make sense of $d_f(z_0,\mu)$ either for our other two example functions, $\Gamma(z+1)$ and $\zeta(s)$, or more generally. We thus turn now to formalising the meaning of the complex derivative $\left(\frac{d}{dz}\right)^{\mu}$ in the LHS of 6 in a manner consistent with properties 7 and 8 but with more general applicability. We do so using Fourier theory, in a manner familiar from the definition of pseudo-differential operators, as follows:

(b)(ii) [Full definition of $\left(\frac{d}{dz}\right)^{\mu}$]: Writing f as the inverse Fourier transform of its Fourier transform, namely

$$g(z) = \frac{1}{2\pi}\iint_{-\infty}^{\infty} g(x)e^{i(z-x)\xi}dx\,d\xi$$

we define $\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{\mu}(g(z))|_{z=z_0}$ by

$$\begin{aligned}
\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{\mu}(g(z))\Big|_{z=z_0} &= \frac{1}{2\pi}\iint_{-\infty}^{\infty}(i\xi)^{\mu}g(x)\mathrm{e}^{i(z_0-x)\xi}\mathrm{d}x\mathrm{d}\xi \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}(i\xi)^{\mu}\mathcal{F}[g](\xi)\mathrm{e}^{iz_0\xi}\mathrm{d}\xi
\end{aligned} \tag{10}$$

With (a)(i), (a)(ii) and (b)(ii), the generalised root identities 6 now have a well-defined meaning for arbitrary complex μ. We have considered their interpretation for $f(z) = \cos(\frac{\pi z}{2})$ and will examine them next for polynomials and for $\Gamma(z+1)$ (and in subsequent papers for $\zeta(s)$). First, however, we make several general observations about these identities and their potential uses.

Observations:(i) [The cases $\mu \in \mathbb{Z}_{\leq 0}$ vs $\mu \in \mathbb{Z}_{>0}$]: While the cases of $\mu \in \mathbb{Z}_{>0}$ give information on the sums of integer powers of shifted reciprocals of roots, 6 gives a much more sensitive relationship between the distribution of these roots and the behaviour of the derivatives of the log of the function.

(ii) [The cases $\mu \in \mathbb{Z}_{\leq 0}$ - asymptotics and geometry]: In particular, for $\mu = 0$, equation 6 should give information about the geometric Césaro *count* of the roots (adding M_i at each point $(z_0 - r_i)$), while for $\mu \in \mathbb{Z}_{<0}$ we get information regarding the geometric Césaro sums of first, second and higher powers of these shifted roots. In particular, taking $Re(\mu)$ sufficiently negative should give information about the asymptotic distribution of the roots near ∞ and the cases $\mu \in \mathbb{Z}_{\leq 0}$ will be of particular interest.

(iii) [The cases $\mu \in \mathbb{Z}_{\leq 0}$ - distributional interpretation]: For $\mu \in \mathbb{Z}_{\leq 0}$, however, it is immediately clear that further care will need to be taken in 6 owing to the poles of Γ in the factor $\frac{1}{\Gamma(\mu)}$ on the LHS. At first glance these appear to make the $d_f(z_0, \mu)$ identically zero whenever $\mu \in \mathbb{Z}_{\leq 0}$ - which would be problematic (e.g. in interpreting the case $\mu = 0$ as yielding a count of roots). Thus, in general, we will need to interpret equation 6 distributionally, and for $\mu \in \mathbb{Z}_{\leq 0}$ the distributional result which we shall rely on in this paper is the following fact (see Hörmander's treatment in [9, pp74]) regarding the function

$$x_+^a := \begin{cases} x^a & , \quad x > 0 \\ 0 & , \quad x < 0 \end{cases}$$

and its normalised counterpart $\chi_+^a := \frac{1}{\Gamma(a+1)} x_+^a$:

Lemma 3: *For $n \in \mathbb{Z}_{\geq 0}$ we have*

$$\chi_+^{-n} = \delta_0^{(n-1)}(x) \tag{11}$$

where δ_0 is the usual delta distribution.

(iv) [The cases $\mu \in \mathbb{Z}_{>0}$ vs $\mu \in \mathbb{Z}_{\leq 0}$ - locality vs non-locality of $\left(\frac{\mathbf{d}}{\mathbf{d}z}\right)^\mu$]: Since the identities 6 give Césaro asymptotic information about the distribution of the roots of f, this makes them a natural tool with which to investigate this asymptotic behaviour. Another way of looking at this is to compare with the case of these identities as expressed in preliminary form in equation 1, where μ was restricted to $\mathbb{Z}_{\geq 0}$.

From an experimental point of view, since for any positive integer μ the RHS of equations 1 or 6 diverges as z_0 approaches any finite root r_i, so the cases of such positive integer μ give a tool for detecting the location of finite roots by looking at the dominant behaviour of these identities as we let z_0 vary. They give little immediate guidance, however, about how the root locations behave asymptotically as they tend to ∞.

By contrast, for $\mu = 0$ the RHS in 6 is clearly insensitive to the location of finite roots, but its geometric Césaro calculation depends critically on the asymptotic location of the roots as they approach ∞; and the same is qualitatively true for $\mu = -1, -2, -3, \ldots$.

In part this in turn reflects the non-locality of the derivatives $\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{\mu}$ on the LHS whenever $\mu \notin \mathbb{Z}_{>0}$, which can briefly be seen by considering an alternative approach to defining such derivatives, as follows. Let T_h be the operator of translation by h (i.e. $T_h[g](z) := g(z+h)$). Since $\frac{\mathrm{d}}{\mathrm{d}z} = \lim_{h\to 0}(\frac{-1}{h})(1 - T_h)$, so formally we have that in general

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{\mu} &= \lim_{h\to 0}\left\{\mathrm{e}^{i\pi\mu}h^{-\mu}(1-T_h)^{\mu}\right\} \\ &= \lim_{h\to 0}\left\{\mathrm{e}^{i\pi\mu}h^{-\mu}\{1-\binom{\mu}{1}T_h+\binom{\mu}{2}T_h^2-\ldots\}\right\}\end{aligned}$$

It is only for $\mu \in \mathbb{Z}_{\geq 1}$ that this series definition truncates and leaves a purely local resulting function. For all other μ the resulting function, $\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{\mu}(g(z))$, depends on $g(z)$ as $z \to \infty$ and so is intrinsically non-local.[4]

(v) [Key Fourier transform results]: In using the Fourier definition of the LHS in 6 we shall use standard Fourier relationships repeatedly, namely

$$\mathcal{F}[xf(x)](\xi) = i\mathcal{F}[f]'(\xi) \quad \& \quad \mathcal{F}[f'(x)](\xi) = i\xi\mathcal{F}[f](\xi)$$

and

$$\mathcal{F}[f(x)e^{iax}](\xi) = F[f](\xi - a) \quad \& \quad \mathcal{F}[f(x+a)](\xi) = \mathcal{F}[f](\xi)e^{ia\xi}\,.$$

Since $\mathcal{F}[1] = 2\pi\delta_0(\xi)$ it follows, on respecting oddness/evenness, that we have

$$\mathcal{F}[\frac{1}{x}] = -2\pi i\tilde{H}_0(\xi) = -2\pi i\cdot(H_0^+(\xi) - \frac{1}{2})$$

and

$$\mathcal{F}[\frac{1}{x^2}] = -2\pi\tilde{H}_0(\xi)\cdot\xi = -2\pi(H_0^+(\xi) - \frac{1}{2})\cdot\xi$$

and in general

$$\mathcal{F}\left[\frac{1}{x^{\rho}}\right] = 2\pi e^{-\frac{i\pi\rho}{2}}\,\frac{\tilde{H}_0(\xi)\xi^{\rho-1}}{\Gamma(\rho)} = 2\pi e^{-\frac{i\pi\rho}{2}}\,\frac{(H_0^+(\xi)-\frac{1}{2})\xi^{\rho-1}}{\Gamma(\rho)} \tag{12}$$

where here, $\tilde{H}_0(\xi)$ is the odd Heaviside function with value $-\frac{1}{2}$ for $\xi < 0$ and value $\frac{1}{2}$ for $\xi > 0$; and $H_0^+(\xi) = \tilde{H}_0(\xi) + \frac{1}{2}$ is the one-sided Heaviside function with corresponding values 0 and 1.

[4] For example for $\mu = -1$ we get $\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{-1} = \lim_{h\to 0}\{-h(1+T_h+T_h^2+\ldots)\}$ which in the limit yields the left Riemann sum definition of $-\int_z^{\infty} g(u)\mathrm{d}u$.

Also, inverting the relationship for Fourier transform of a derivative and applying it to the formula for $\mathcal{F}[\frac{1}{x}]$ above, we have

$$\mathcal{F}[\ln x] = -2\pi \frac{\tilde{H}_0(\xi)}{\xi} = -2\pi \frac{(H_0^+(\xi) - \frac{1}{2})}{\xi} \tag{13}$$

and

$$\mathcal{F}[x \ln x - x] = 2\pi i \, \frac{\tilde{H}_0(\xi)}{\xi^2} = 2\pi i \, \frac{(H_0^+(\xi) - \frac{1}{2})}{\xi^2} \quad \text{\& so on} \tag{14}$$

with corresponding results for higher anti-derivatives on the LHS.[5] It is these latter identities that will be used most directly in what follows.

In calculating on the derivative side of the root identities 6, note that $H_0^+(\xi) \cdot \xi^a = \xi_+^a$ so that, for $a \in \mathbb{Z}_{<0}$, the first term in the brackets in these expressions leads naturally to use of the Hörmander result 11. The second half of these expressions, $\frac{1}{2}\xi^a$, leads, under inverse Fourier transform, either to δ-function type contributions (for $a \in \mathbb{Z}_{\geq 0}$) which may be ignored for $z_0 \neq 0$; or to Heaviside-type contributions in z_0 (for $a \in \mathbb{Z}_{<0}$), whose finite values for any z_0 cancel to 0 under the factor $\frac{1}{\Gamma(\mu)}$ on the derivative side when $\mu \in \mathbb{Z}_{\leq 0}$.

As such, for our calculations when $\mu \in \mathbb{Z}_{\leq 0}$ in the root identities for Γ in the next section, we may effectively ignore these second-half contributions and elide the distinction between $\tilde{H}_0(\xi) \cdot \xi^a$ and ξ_+^a.

Having verified in subsection 4.3.1 that $f(z) = \cos(\frac{\pi z}{2})$ satisfies the full generalised root identities 6 for arbitrary $\mu \in \mathbb{C}$, and having now formalised how to make sense of both the root and derivative sides of these identities in general, let us now also assess them for the other example functions considered previously.

4.3 Further example cases for the full generalised root identities

Before turning to the case of $\Gamma(z+1)$, let us first re-consider the case of polynomials, where we started. It is trivial that these satisfy the generalised root identities for $\mu \in \mathbb{Z}_{\geq 1}$, but it is not self-evident that this continues to hold for arbitrary $\mu \in \mathbb{C}$, and indeed demonstrating that it does is non-trivial.

4.3.1 The generalised root identities for $f(z) = z$ and for polynomials

The case of $f(z) = z$: In this case f has a single root of multiplicity 1 at $z = 0$ and so the root-side of the generalised root identities is given by

$$r(z_0, \mu) = \frac{\mathrm{e}^{i\pi\mu}}{z_0^{\mu}} \tag{15}$$

[5] Technically in fact $\mathcal{F}[\ln x] = -2\pi \frac{\tilde{H}_0(\xi)}{\xi} - 2\pi\gamma\delta_0(\xi)$ with the extra term arising from the constant of integration. However, we ignore the additional $\delta_0(\xi)$ term here since it will not contribute to the derivative side of the root identities at $\mu \in \mathbb{Z}_{\leq 0}$ in any of our remaining calculations in this paper, on account of the factor $\frac{1}{\Gamma(\mu)}$ whose denominator diverges at such μ.

We now consider the derivative side in several steps.

(i) [The case of $\mu \in \mathbb{Z}_{>0}$]: When $\mu = 1$ the derivative side is

$$d(z_0, 1) = \frac{-1}{\Gamma(1)} \frac{\mathrm{d}}{\mathrm{d}z}(\ln z)\Big|_{z=z_0} = \frac{-1}{z_0} = r(z_0, 1) \tag{16}$$

and by direct differentiation it is then trivial to see that we likewise have

$$d(z_0, \mu) = \frac{\mathrm{e}^{i\pi\mu}}{z_0^{\mu}} = r(z_0, \mu) \qquad \text{for all } \mu \in \mathbb{Z}_{>0} \tag{17}$$

so that certainly $f(z) = z$ satisfies the generalised root identities for $\mu \in \mathbb{Z}_{>0}$.[6]

(ii) [The case of $\mu \in \mathbb{Z}_{\leq 0}$:] Recalling the definition of the derivative side, that

$$d_f(z_0, \mu) = -\frac{1}{\Gamma(\mu)} \frac{1}{2\pi} \int_{-\infty}^{\infty} (i\xi)^{\mu} \mathcal{F}[\ln f](\xi) \mathrm{e}^{iz_0\xi} \mathrm{d}\xi \tag{18}$$

and that $\mathcal{F}[\ln z](\xi) = -2\pi \frac{\tilde{H}_0(\xi)}{\xi}$, we can calculate the derivative side via distributional calculations as outlined in section 4.2. For $\mu = 0$ we have

$$d(z_0, 0) = \frac{1}{\Gamma(0)} \int_{-\infty}^{\infty} \frac{\tilde{H}_0(\xi)}{\xi} \mathrm{e}^{iz_0\xi} \mathrm{d}\xi = \delta_0(\xi)[\mathrm{e}^{iz_0\xi}] = 1 = r(z_0, 0) \tag{19}$$

while for $\mu = -1$ we have

$$d(z_0, -1) = \frac{-i}{\Gamma(-1)} \int_{-\infty}^{\infty} \frac{\tilde{H}_0(\xi)}{\xi^2} \mathrm{e}^{iz_0\xi} \mathrm{d}\xi = -i \cdot \delta_0'(\xi)[\mathrm{e}^{iz_0\xi}] = -z_0 = r(z_0, -1) \tag{20}$$

and similarly for $\mu = -2, -3, \ldots$, so that $f(z) = z$ also satisfies the generalised root identities for $\mu \in \mathbb{Z}_{\leq 0}$.

(iii) [The case of $\mu \in (0, 1)$]: Assume initially that z_0 is also real and positive. Then the derivative side, $d(z_0, \mu)$ is given by

$$d(z_0, \mu) = \frac{1}{\Gamma(\mu)} \int_{-\infty}^{\infty} (i\xi)^{\mu} \frac{\tilde{H}_0(\xi)}{\xi} \mathrm{e}^{iz_0\xi} \mathrm{d}\xi \tag{21}$$

$$= \frac{\mathrm{e}^{i\frac{\pi}{2}\mu}}{\Gamma(\mu)} \int_{-\infty}^{\infty} \tilde{H}_0(\xi) \xi^{\mu-1} (\cos(z_0\xi) + i\sin(z_0\xi)) \mathrm{d}\xi$$

$$= \frac{\mathrm{e}^{i\frac{\pi}{2}\mu}}{2\Gamma(\mu)} \left\{ \begin{array}{c} (1 - \mathrm{e}^{i\pi(\mu-1)}) \int_0^{\infty} \xi^{\mu-1} \cos(z_0\xi) \mathrm{d}\xi \\ +i \cdot (1 + \mathrm{e}^{i\pi(\mu-1)}) \int_0^{\infty} \xi^{\mu-1} \sin(z_0\xi) \mathrm{d}\xi \end{array} \right\} \tag{22}$$

[6]Note in passing that these results can be derived via integration from the Fourier definition of the derivative side using Césaro methods, but there is obviously no need when $\mu \in \mathbb{Z}_{>0}$.

on recalling the definition of $\tilde{H}_0(\xi)$. Changing variables in these integrals by letting $v = (z_0\xi)^\mu$ and noting that $\mathrm{d}v = \mu z_0^\mu \xi^{\mu-1}\mathrm{d}\xi$, it follows that

$$d(z_0,\mu) = \frac{\mathrm{e}^{i\frac{\pi}{2}\mu}}{2\Gamma(\mu+1)} \left\{ \begin{array}{c} (1+\mathrm{e}^{i\pi\mu}) \int_0^\infty \cos(v^{\frac{1}{\mu}})\mathrm{d}v \\ \\ +i\cdot(1-\mathrm{e}^{i\pi\mu}) \int_0^\infty \sin(v^{\frac{1}{\mu}})\mathrm{d}v \end{array} \right\} \cdot z_0^{-\mu} \quad . \tag{23}$$

But recall the generalised Fresnel integral identity that

$$\int_0^\infty \sin(x^a)\mathrm{d}x = \frac{\Gamma(\frac{1}{a})\sin(\frac{\pi}{2a})}{a} \quad , \tag{24}$$

defined initially for $a > 1$ and then extended by analytic continuation.[7] From this it readily follows also that

$$\int_0^\infty \cos(x^a)\mathrm{d}x = \frac{\Gamma(\frac{1}{a})\cos(\frac{\pi}{2a})}{a} \quad . \tag{25}$$

Combining equations 24 and 25 it follows that we have

$$\begin{aligned} d(z_0,\mu) &= \frac{\mathrm{e}^{i\frac{\pi}{2}\mu}\mu\Gamma(\mu)}{2\Gamma(\mu+1)} \Big\{(1+\mathrm{e}^{i\pi\mu})\cos(\frac{\pi\mu}{2}) + i\cdot(1-\mathrm{e}^{i\pi\mu})\sin(\frac{\pi\mu}{2})\Big\} \cdot z_0^{-\mu} \\ &= \frac{\mathrm{e}^{i\frac{\pi}{2}\mu}}{4} \left\{(1+\mathrm{e}^{i\pi\mu})(\mathrm{e}^{i\frac{\pi}{2}\mu} + \mathrm{e}^{-i\frac{\pi}{2}\mu}) + (1-\mathrm{e}^{i\pi\mu})(\mathrm{e}^{i\frac{\pi}{2}\mu} - \mathrm{e}^{-i\frac{\pi}{2}\mu})\right\} \cdot z_0^{-\mu} \\ &= \frac{\mathrm{e}^{i\pi\mu}}{z_0^\mu} = r(z_0,\mu) \quad . \end{aligned} \tag{26}$$

Thus we see that $f(z) = z$ also satisfies the generalised root identities for arbitrary $\mu \in (0,1)$ and $z_0 \in \mathbb{R}_{>0}$.

(iv) [Extension to arbitrary $\mu, z_0 \in \mathbb{C}$]: Keeping $z_0 \in \mathbb{R}_{>0}$ initially, we can then extend from $\mu \in (0,1)$ to all of $\mu \in \mathbb{R} \setminus \mathbb{Z}$ by a combination of differentiation under the integral w.r.t z_0 and homogeneity arguments (which guarantee there are no constants of integration).

We thus have now verified that $f(z) = z$ satisfies the generalised root identities for arbitrary $\mu \in \mathbb{R}$ and $z_0 \in \mathbb{R}_{>0}$. By unique analytic continuation they therefore continue to hold first for arbitrary $\mu \in \mathbb{C}$ and $z_0 \in \mathbb{R}_{>0}$, and finally also for arbitrary $\mu \in \mathbb{C}$ and $z_0 \in \mathbb{C}$, except for a branch cut from $z_0 = -\infty$ to $z_0 = 0$ when $\mu \notin \mathbb{Z}$.

The Case of Polynomials: Since $f(z) = z$ satisfies the generalised root identities for arbitrary $\mu, z_0 \in \mathbb{C}$ it follows readily from equation 18 and elementary properties of the Fourier transform (in particular that $\mathcal{F}[f(x+a)](\xi) = \mathcal{F}[f](\xi)\mathrm{e}^{ia\xi}$) that the generalised root identities are also satisfied for arbitrary $z_0, \mu \in \mathbb{C}$ by $f(z) = z - a$.

[7] When $a = 2$ this reduces to the classical Fresnel integral from optics $\int_0^\infty \sin(x^2)\mathrm{d}x = \sqrt{\frac{\pi}{8}}$

But then it also follows immediately from the fact that $\ln((z-r_1)(z-r_2)) = \ln(z-r_1) + \ln(z-r_2)$ and the obvious additivity of the root side, that the generalised root identities must also be satisfied for arbitrary $z_0, \mu \in \mathbb{C}$ by any polynomial $p(z)$. For any given $p(z)$, the structure of the branch cuts in z_0 when $\mu \notin \mathbb{Z}$ (corresponding to the branch cut on $z_0 \in (-\infty, 0)$ for $f(z) = z$ replicated for each factor) is slightly more complicated, but there are no branch cut complications for generic z_0, and in particular for all z_0 in a half-plane to the right of the root with the most positive real part, which is all we require in subsequent arguments.

4.3.2 The generalised root identities for $\Gamma(z+1)$

The root side: For $\Gamma(z+1)$ on the root side of equation 6 we clearly have at once that

$$\begin{aligned} r_\Gamma(z_0,\mu) &= e^{i\pi\mu} \sum_{\{z_0-roots\, r_i\}} \frac{M_i}{(z_0-r_i)^\mu} \\ &= -e^{i\pi\mu} \sum_{n=1}^{\infty} (z_0+n)^{-\mu} = -e^{i\pi\mu}\zeta_H(z_0,\mu) \end{aligned} \tag{27}$$

which is classically convergent for $Re(\mu) > 1$ and geometrically Césaro convergent for general μ (renormalisation being required as discussed when $\mu = 1$). For example, using the Euler-McLaurin sum formula and the same sort of reasoning as used in [I], when $0 < \mu < 1$ we have $r_\Gamma(z_0,\mu)$ given by

$$r_\Gamma(z_0,\mu) = -\mathrm{e}^{i\pi\mu} \lim_{k\to\infty} \left\{ \sum_{j=1}^{k} \frac{1}{(z_0+j)^\mu} - \frac{(z_0+k)^{1-\mu}}{1-\mu} \right\} \tag{28}$$

while for $-1 < \mu < 0$ we have

$$r_\Gamma(z_0,\mu) = -\mathrm{e}^{i\pi\mu} \lim_{k\to\infty} \left\{ \sum_{j=1}^{k} \frac{1}{(z_0+j)^\mu} - \frac{(z_0+k)^{1-\mu}}{1-\mu} - \frac{1}{2}(z_0+k)^{-\mu} \right\} \tag{29}$$

and so on. For $\mu \notin \mathbb{Z}$ there is a single branch cut on $z_0 \in (-\infty, -1)$ (with branch points at $-1, -2, -3, \ldots$).

When $z_0 = 0$ equation 27 reduces to the zeta function

$$r_\Gamma(0,\mu) = -\mathrm{e}^{i\pi\mu}\zeta(\mu) \tag{30}$$

and since $\zeta_H(z_0,\mu) = \zeta(\mu) - \sum_{j=1}^{z_0} j^{-\mu}$, so for example at $\mu = 0, -1, -2, -3, \ldots$ we get

$$\begin{aligned} r_\Gamma(z_0, 0) &= z_0 + \frac{1}{2} \\ r_\Gamma(z_0, -1) &= -\frac{1}{2}z_0^2 - \frac{1}{2}z_0 - \frac{1}{12} \end{aligned}$$

$$
\begin{aligned}
r_\Gamma(z_0,-2) &= \frac{1}{3}z_0^3+\frac{1}{2}z_0^2+\frac{1}{6}z_0 \\
r_\Gamma(z_0,-3) &= -\frac{1}{4}z_0^4-\frac{1}{2}z_0^3-\frac{1}{4}z_0^2+\frac{1}{120} \quad .
\end{aligned}
\tag{31}
$$

In general for $n \in \mathbb{Z}_{>0}$ we have

$$
r_\Gamma(z_0,-n) = (-1)^n\{b_{n+1}(z_0)-\zeta(-n)\} = -\frac{1}{n+1}B_{n+1}(-z_0) \tag{32}
$$

where $b_{n+1}(z_0) := \sum_{j=1}^{z_0} j^n$ and B_{n+1} is the $(n+1)^{st}$ Bernoulli polynomial.

The derivative side: On the derivative side, noting $\left(\frac{d}{dz}\right)^\mu = \left(\frac{d}{dz}\right)^{\mu-1}\left(\frac{d}{dz}\right)$ and recalling the identity in equation 3, we have

$$
\begin{aligned}
d_\Gamma(z_0,\mu) &= -\frac{1}{\Gamma(\mu)}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^\mu (\ln(\Gamma(z+1)))|_{z=z_0} \\
&= \frac{1}{2\pi}\frac{1}{\Gamma(\mu)}\iint_{-\infty}^{\infty}(i\xi)^{\mu-1}\left\{\sum_{n=1}^{\infty}\left(\frac{1}{n+x}-\frac{1}{n}\right)+\gamma\right\}\mathrm{e}^{i(z_0-x)\xi}\mathrm{d}x\mathrm{d}\xi
\end{aligned}
$$

For all except $\mu=1$ we can omit the $\delta_0(\xi)$ terms arising from the $\frac{1}{n}$ and γ terms here. Thus, by the Fourier identities canvassed in (v), we obtain that

$$
\begin{aligned}
d_\Gamma(z_0,\mu) &= \frac{1}{\Gamma(\mu)}\int_{-\infty}^{\infty}(i\xi)^{\mu-1}\left\{\sum_{n=1}^{\infty}-i\tilde{H}_0(\xi)e^{in\xi}\right\}e^{iz_0\xi}\,d\xi \\
&= \frac{1}{\Gamma(\mu)}\int_{-\infty}^{\infty}\tilde{H}_0(\xi)\,(i\xi)^{\mu-1}\left(i\frac{e^{i\xi}}{e^{i\xi}-1}\right)e^{iz_0\xi}\,d\xi \\
&= -\frac{ie^{i\frac{\pi}{2}\mu}}{\Gamma(\mu)}\int_{-\infty}^{\infty}\tilde{H}_0(\xi)\,\xi^{\mu-2}\left\{\begin{array}{c} 1-\left[\frac{(i\xi)}{2!}+\frac{(i\xi)^2}{3!}+\dots\right]+ \\ \\ \left[\frac{(i\xi)}{2!}+\frac{(i\xi)^2}{3!}+\dots\right]^2-\dots\end{array}\right\}e^{i(z_0+1)\xi}\,d\xi
\end{aligned}
\tag{33}
$$

In the case where $\mu \in \mathbb{Z}_{\leq 0}$, we may further ignore the contributions from the $\frac{1}{2}$ term in $\tilde{H}_0(\xi) = H_0^+(\xi)-\frac{1}{2}$ as previously discussed, and rewrite this as

$$
\begin{aligned}
d_\Gamma(z_0,\mu) &= \frac{i}{\Gamma(\mu)}\int_0^{\infty}(i\xi)^{\mu-2}\mathrm{e}^{i(z_0+1)\xi}\left\{\begin{array}{c} 1-\left[\frac{(i\xi)}{2!}+\frac{(i\xi)^2}{3!}+\dots\right]+ \\ \left[\frac{(i\xi)}{2!}+\frac{(i\xi)^2}{3!}+\dots\right]^2-\dots\end{array}\right\}\mathrm{d}\xi \\
&= \frac{i}{\Gamma(\mu)}\int_0^{\infty}(i\xi)^{\mu-2}\left\{\begin{array}{c} 1+i\left[(z_0+\frac{1}{2})\right]\xi+\left[-\frac{1}{2}z_0^2-\frac{1}{2}z_0-\frac{1}{12}\right]\xi^2 \\ \\ +(-i)\left[\frac{1}{6}z_0^3+\frac{1}{4}z_0^2+\frac{1}{12}z_0\right]\xi^3 \\ \\ +\left[\frac{1}{24}z_0^4+\frac{1}{12}z_0^3+\frac{1}{24}z_0^2+\frac{1}{120}\right]\xi^4+\dots\end{array}\right\}\mathrm{d}\xi
\end{aligned}
$$

Comparing $r_\Gamma(z_0, \mu)$ and $d_\Gamma(z_0, \mu)$: Case (i) [$\mu \in \mathbb{Z}_{\leq 0}$]: Now suppose $\mu \in \mathbb{Z}_{\leq 0}$. Working distributionally as discussed earlier, recall that for $\mu \in \mathbb{Z}_{\leq 0}$ we have $\frac{\xi_+^{\mu-2}}{\Gamma(\mu-1)} = \delta_0^{(1-\mu)}(\xi)$ (by equation 11). Thus for $\mu = 0$ we have immediately that $d_\Gamma(z_0, 0) = z_0 + \frac{1}{2}$, in agreement with the first result from 31. Similarly, for $\mu = -1, -2$ and -3, after disentangling the impact of the $\frac{1}{\mu-1}$ factor and the factorial arising from differentiation, we get agreement with the formulae in 31 (each arising as a scalar multiple of the appropriate coefficient of $\xi_+^{1-\mu}$ in the bracket); and in general

$$\forall \mu \in Z_{\leq 0} \qquad d_\Gamma(z_0, \mu) = r_\Gamma(z_0, \mu)$$

so that $\Gamma(z + 1)$ does satisfy the generalised root identities 6 for all $\mu \in \mathbb{Z}_{\leq 0}$.

Case (ii) [$\mu \in \mathbb{R} \setminus \mathbb{Z}$]: As for the case when $\mu \in \mathbb{R} \setminus \mathbb{Z}$, the working in section 4.3.1, in particular equations 15 and 21, has shown that

$$\frac{e^{i\frac{\pi}{2}\rho}}{\Gamma(\rho)} \int_{-\infty}^{\infty} \tilde{H}_0(\xi)\, \xi^{\rho-1} e^{iz_0\xi} d\xi = \frac{e^{i\pi\rho}}{z_0^\rho} \tag{34}$$

for arbitrary $\rho \in \mathbb{R} \setminus \mathbb{Z}$ and $z_0 \notin (-\infty, 0)$.

It follows in equation 33 that, for arbitrary $\mu \in \mathbb{R} \setminus \mathbb{Z}$ and $z_0 \notin (-\infty, -1)$, we have, after simplification,

$$d_\Gamma(z_0, \mu) = \left\{ \begin{array}{c} \frac{1}{(\mu-1)} \mathrm{e}^{i\pi(\mu-1)} \frac{1}{(z_0+1)^{(\mu-1)}} - \frac{1}{2} \mathrm{e}^{i\pi\mu} \frac{1}{(z_0+1)^\mu} \\ + \frac{\mu}{12} \mathrm{e}^{i\pi(\mu+1)} \frac{1}{(z_0+1)^{(\mu+1)}} \\ - \frac{\mu(\mu+1)(\mu+2)}{720} \mathrm{e}^{i\pi(\mu+3)} \frac{1}{(z_0+1)^{(\mu+3)}} + \ldots \end{array} \right\} \tag{35}$$

$$= -\frac{\mathrm{e}^{i\pi\mu}}{(\mu-1)} \frac{1}{(z_0+1)^{(\mu-1)}} \left\{ \begin{array}{c} 1 + \frac{1}{2} \frac{\mu-1}{(z_0+1)} \\ + \frac{(\mu-1)\mu B_2}{2!(z_0+1)^2} \\ + \frac{(\mu-1)\mu(\mu+1)(\mu+2)B_4}{4!(z_0+1)^4} + \ldots \end{array} \right\} \tag{36}$$

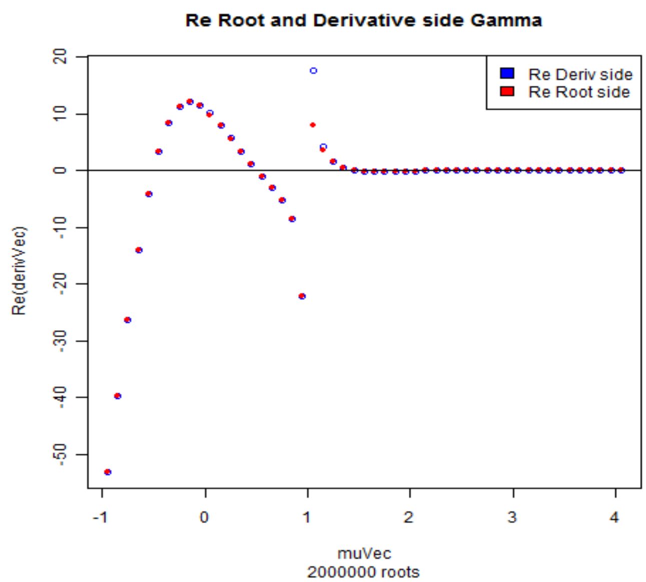
as an Euler-McLaurin-style asymptotic expansion for the derivative side.[8]

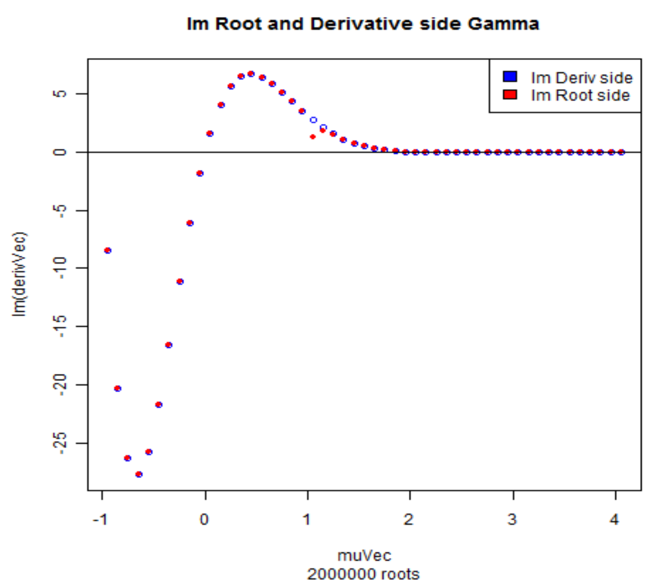
Using equations 27-36 we may then write code to numerically check the generalised root identities for $\Gamma(z + 1)$ when $\mu \notin \mathbb{Z}$. In particular, for any given $\mu \notin \mathbb{Z}$ we can evaluate $d_\Gamma(z_0, \mu)$ to any desired degree of accuracy for $Re(z_0)$ sufficiently large by using sufficient terms in equation 35; and verifying that

[8] In fact this is precisely the Euler-McLaurin asymptotic expansion for $\mathrm{e}^{i\pi\mu}\zeta_H(z_0, \mu)$, in agreement with $r_\Gamma(z_0, \mu)$ in equation 27. However, we think it is valuable to validate the generalised root identities for Γ more concretely in this section, via numerical testing, and so we focus on that instead here.

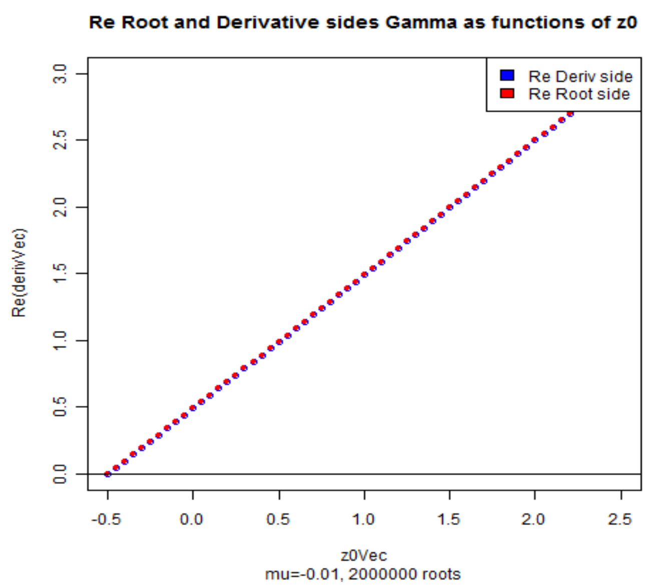
$d_\Gamma(z_0, \mu) = r_\Gamma(z_0, \mu)$ on such a half-plane in z_0 (sufficiently far to the right for the given μ) suffices to verify their agreement for all z_0 by analytic continuation for the given μ.

Numerical investigations implementing equations 27-36 have been implemented in R-code by Dana Pascovici. It has been found that $\Gamma(z+1)$ does indeed appear to satisfy the generalised root identities for arbitrary $z_0, \mu \in \mathbb{C}$ (with branch cut on $z_0 \in (-\infty, -1)$ when $\mu \notin \mathbb{Z}$). For example, consider the following figures:





These pictures show agreement between real and imaginary parts of $d_\Gamma(z_0, \mu)$ and $r_\Gamma(z_0, \mu)$ for a range of μ-values from -1 to 4 for $z_0 = 10.381$ using $2,000,000$ roots to approximate $r_\Gamma(z_0, \mu)$ and truncating the expression for $d_\Gamma(z_0, \mu)$ at $\frac{1}{(z_0+1)^{(\mu+5)}}$. The following figure also shows how, for μ small ($\mu = -0.01$), the function $r_\Gamma(z_0, -0.01)$ closely approaches the limiting function $r_\Gamma(z_0, 0)$ which we know from equation 31 is given by $r_\Gamma(z_0, 0) = z_0 + \frac{1}{2}$.



The R-script for Gamma is made available with this paper and further numerical tests may be performed as desired.

Thus, overall, we see that $\Gamma(z + 1)$ satisfies the generalised root identities 6 for arbitrary z_0 and $\mu \in \mathbb{R} \setminus \{1\}$, and hence also for arbitrary $\mu \in \mathbb{C} \setminus \{1\}$ by analytic continuation. In fact, just as before, it is easy to adapt the above arguments to verify that this remains true for $\Gamma(az + b)$ for any $a, b \in \mathbb{C}$.

Final comments: **(i)** While we have focused above on validating the generalised root identities for Γ *numerically* for $\mu \in \mathbb{R} \setminus \mathbb{Z}$, a formal proof could also readily be supplied. This could be done using our earlier equation for $\frac{d^2}{dz^2} \ln(\Gamma(z + 1))$, reversing order of summation and integration, and judiciously applying result 12 along with the basic properties of Fourier transforms. Alternatively, as noted before, we could compare the Euler-McLaurin-style asymptotic expansion in equation 36 with the known expansion for $\zeta_H(z_0, \mu)$ and invoke equation 27. In keeping with our preference for practical and calculational byways in this paper, however, we omit any such proof, along with any discussion of the technical issues that attach to it.

(ii) One point is nevertheless worth noting - namely that it is not just on the root side that generalised Césaro methods can be employed. Indeed, many calculations on the derivative side (for example of the integrals involved in our Fourier-theoretic definitions there) can also be handled successfully if we simply embrace a generalised Césaro approach wholeheartedly and unapologetically. We shall see some of this in the remaining papers in this series, and then also in a subsequent series of papers.

4.4 Stirling's theorem from the root identities for Γ for $\mu \in \mathbb{Z}_{\leq 0}$

To conclude our analysis of $\Gamma(z+1)$ we now illustrate the promised power of the root identities for $\mu \in \mathbb{Z}_{\leq 0}$ to give information about asymptotic behaviour. We do this by constructively deducing Stirling's theorem - giving the asymptotic behaviour of $\Gamma(z+1)$ as $z \to \infty$ - from successive consideration of the root identities for $\mu = 0, -1, -2, \ldots$.

Stirling's Theorem: *As $z \to +\infty$ we have*

$$\Gamma(z+1) = \sqrt{2\pi}\, z^{z+\frac{1}{2}} e^{-z} e^{J(z)} \tag{37}$$

where $J(z)$ has the asymptotic expansion

$$J(z) = \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n-1)\cdot 2n} \frac{1}{z^{2n-1}} = \frac{1}{12}\frac{1}{z} - \frac{1}{360}\frac{1}{z^3} + \ldots \tag{38}$$

Equivalently, as $z \to +\infty$, we have asymptotically that

$$\ln(\Gamma(z+1)) = (z+\frac{1}{2})\ln z - z + \frac{1}{2}\ln 2\pi + J(z) \quad . \tag{39}$$

Derivation: To deduce this, consider first the generalised root identity for $\mu = 0$. By 31 we have seen (from a Césaro count of roots on the RHS) that this means

$$\lim_{\mu\to 0} \frac{-1}{\Gamma(\mu)} \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{\mu} (\ln(\Gamma(z+1)))\Big|_{z=z_0} = z_0 + \frac{1}{2} \tag{40}$$

As we move through $\mu = -1, -2, \ldots$ the corresponding polynomials in equation 31 are successive integrals of this (since each arises from an additional application of $\left(\frac{d}{dz}\right)^{-1}$, i.e. integration, on the LHS). The only new information each time is the value of the integration constant, namely $(-1)^{\mu-1}\zeta(\mu)$ (by (32)).

Our requirement is thus **(a)** to understand what terms must occur in the expansion for $\ln(\Gamma(z+1))$ as $z \to \infty$ in order for the terms z_0 and $\frac{1}{2}$ to arise on the LHS in equation 40 when $\mu = 0$; and then **(b)** to identify how the successive integration constants just mentioned can be made to appear on the derivative side of the root identities 6 when $\mu = -1, -2, \ldots$ by the inclusion of further terms; all **(c)** without disturbing the identities already examined.

Step (a): Consider the definition in equation 10. We see that integer powers of z_0 will arise when the action of $\frac{-1}{\Gamma(\mu)}\frac{1}{2\pi}\int_{-\infty}^{\infty}(i\xi)^\mu\mathcal{F}[g](\xi)\bullet d\xi$ on $e^{iz_0\xi}$ consists of differentiation to some integer order. In light of 11 this means we require

$$\frac{-1}{\Gamma(\mu)}\,\xi^\mu\mathcal{F}[g](\xi)=\chi_+^{-n}(\xi)=\frac{1}{\Gamma(1-n)}\frac{H_0(\xi)}{\xi_+^n}$$

for some $n\in\mathbb{Z}_{>0}$.

Since we are considering only $\mu\in\mathbb{Z}_{\le 0}$ we thus require $\mathcal{F}[g](\xi)$ itself to be of the form $C\frac{\tilde{H}_0(\xi)}{\xi^\rho}$ for some suitable C and ρ (ignoring the contribution from the $\frac{1}{2}$ in $\tilde{H}_0(\xi)=H_0^+(\xi)-\frac{1}{2}$ as usual).

In particular, for $\mu=0$, it is clear that in order to get a constant term $\frac{1}{2}$ we need to have $\mathcal{F}[g](\xi)=\frac{1}{2}\frac{\tilde{H}_0(\xi)}{\xi}$ (so that we get $\frac{1}{2}\chi_+^{-1}$, i.e. $\frac{1}{2}\delta_0(\xi)$, acting on $e^{iz_0\xi}$); while to get a term z_0 we need to have $\mathcal{F}[g](\xi)=\frac{\tilde{H}_0(\xi)}{\xi^2}$ (so that we get χ_+^{-2}, i.e. $-\delta_0'(\xi)$, acting on $e^{iz_0\xi}$).

But, comparing with equation 12, it is clear that this means we need to take $g(x)=(x\ln x-x)+\frac{1}{2}\ln x$ and so $(z\ln z-z)+\frac{1}{2}\ln z$ must appear in the expansion for $\ln(\Gamma(z+1))$.

Step (b): Turning next to the $\mu=-1$ identity, these terms in turn integrate to give the terms $-\frac{1}{2}z_0^2-\frac{1}{2}z_0$ in equation 31, leaving only the presence of the $\frac{-1}{12}$ term to account for. To understand this, we recall how previously we used terms e^{az^n} to "heal" obstructions for root identities with $\mu\in\mathbb{Z}_{>0}$. Here it is natural instead to consider terms of the form $e^{\frac{a}{z^n}}$ to "heal" corresponding obstructions when $\mu\in\mathbb{Z}_{\le 0}$.

Consider $e^{\frac{a}{z}}$ first. On taking logs, this becomes $g(z)=\frac{a}{z}$ and, by equation 12, $\mathcal{F}[g](\xi)=-2\pi ia\tilde{H}_0(\xi)$. Thus, when $\mu=-1$, $(i\xi)^\mu\mathcal{F}[g](\xi)=-\frac{2\pi a}{\xi_+}$ (eliding $\tilde{H}_0(\xi)$ and $H_0^+(\xi)$ as usual) and on writing $\frac{1}{\Gamma(\mu)}=\frac{\mu}{\Gamma(\mu+1)}$ we see from equation 11 that we will get a contribution of $-a$ from this term in the root identity (arising from $-a\delta_0(\xi)$ acting on $e^{iz_0\xi}$). In order to match the required $\frac{-1}{12}$ we take $a=\frac{1}{12}$. We thus include in the asymptotic expansion as $z\to\infty$ an extra factor of $e^{\frac{1}{12}\frac{1}{z}}$ in $\Gamma(z+1)$ (or $\frac{1}{12}\frac{1}{z}$ in $\ln(\Gamma(z+1))$), in order to satisfy the $\mu=-1$ root identity.

Importantly, note that this extra term does not disturb the existing identity at $\mu=0$ since there it leads to a finite Césaro integral which is cancelled to 0 by the $\frac{1}{\Gamma(0)}$ factor in the $\mu=0$ identity.

So far, from the $\mu=0$ and $\mu=-1$ identities, we have derived contributions $z^{z+\frac{1}{2}}e^{-z}e^{\frac{1}{12}\frac{1}{z}}$ in $\Gamma(z+1)$. Working inductively, suppose the extra terms up to degree $-n+1$ have been derived from the root identities for $\mu=-2,\dots,-n+1$. Then, in the same way, for $\mu=-n$ these terms will integrate to yield the polynomial $(-1)^nb_{n+1}(z_0)$ in the expression 32 for the $\mu=-n$ root identity (after adjusting for the change in $\frac{1}{\Gamma(\mu)}$ factor on the derivative side), and we need to introduce a further term $e^{\frac{a}{z^n}}$ in order to match the new constant term $-(-1)^n\zeta(-n)$ which is all that remains unaccounted for.

But taking logs yields $g(z) = \frac{a}{z^n}$ so $\mathcal{F}[g](\xi) = 2\pi a e^{-i\frac{\pi}{2}n}\frac{\tilde{H}_0(\xi)\xi^{n-1}}{\Gamma(n)}$ and thus $-\frac{1}{\Gamma(\mu)}\frac{1}{2\pi}(i\xi)^\mu\mathcal{F}[g](\xi) = -a\cdot\frac{n}{\Gamma(0)}\frac{1}{\xi_+} = -a\cdot n\cdot\delta_0(\xi)$, after simplifying the two Γ terms by cancellation (or alternatively using the functional identity for the Gamma function) and retaining only the $H_0^+(\xi)$ term from $\tilde{H}_0(\xi)$ as usual. Acting on $e^{iz_0\xi}$ we thus need to take $a = \frac{(-1)^n\zeta(-n)}{n}$ in order to match the required constant term.

When n is even (i.e. $\mu = -2, -4, \ldots$) we are at a trivial zero of ζ and $a = 0$, so that there are no even order reciprocal terms in the asymptotic expansion for $J(z)$. When $n = 2k-1$ is odd we know that $\zeta(-(2k-1)) = -\frac{B_{2k}}{2k}$ so that $a = \frac{B_{2k}}{2k\cdot(2k-1)}$ and we have to include a term $e^{\frac{B_k}{2k\cdot(2k-1)}\frac{1}{z^{2k-1}}}$ in our asymptotic expansion for $\Gamma(z+1)$. And, again, each new term included does not disturb our existing identities for $\mu = -n+1, -n+2, \ldots, -1, 0$ - since there the $e^{\frac{a}{z^n}}$ term leads to finite Césaro integrals on the derivative side which cancel to 0 against the $\frac{1}{\Gamma(j)}$ factor in the $\mu = j$ identity, $-n+1 \leq j \leq 0$.

Successive consideration of the root identities for $\mu \in \mathbb{Z}_{\leq 0}$ thus leads us constructively to deduce that

$$\Gamma(z+1) = C\, z^{z+\frac{1}{2}}\mathrm{e}^{-z}\mathrm{e}^{J(z)}$$

exactly as in equations 37 and 38, and it remains only to show that $C = \sqrt{2\pi}$.

But, letting $z = k \to \infty$ through the positive integers, taking logs and noting that $J(k) \to 0$ as $k \to \infty$ we get

$$\ln C = \lim_{k\to\infty}\left(\sum_{j=1}^{k}\ln j - \left[(k+\frac{1}{2})\ln k - k\right]\right) \quad .$$

This last expression represents a continuous generalised Césaro expression for $-\zeta'(0)$ (see e.g. discussion in [II]); thus $\ln C = \frac{1}{2}\ln(2\pi)$ and $C = \sqrt{2\pi}$ as required.

Step (c): This completes the constructive derivation of the expression in 37 for $\Gamma(z+1)$ as $z \to \infty$, purely from the root identities for $\mu \in \mathbb{Z}_{\leq 0}$. To complete the proof of Stirling's theorem it remains only to verify also that the inclusion of these extra factors of the form $\mathrm{e}^{\frac{a}{z^n}}$ has not disturbed our previous verification that $\Gamma(z+1)$ satisfies the root identities for $\mu \in \mathbb{Z}_{>0}$, i.e. that the expression $\sqrt{2\pi}\, z^{z+\frac{1}{2}}\mathrm{e}^{-z}\mathrm{e}^{J(z)}$ continues to satisfy all these root identities for $\Gamma(z+1)$ without requiring the healing of any new obstructions.

In one sense, this is just the same observation for non-positive powers that we have previously noted for positive powers - namely that inclusion of a term $\frac{a}{z^n}$ in $\ln(\Gamma(z+1))$ has no effect on $d_\Gamma(z_0, \mu)$ for any μ other than $\mu = -n$. But the argument demonstrating this is less straightforward in this case than for positive powers, owing to the need to invoke Fourier and distributional results to perform it. As such it is worth working through it at least this one time.

For $\mu = 1$ we immediately encounter a problem from the term $z^z e^{-z}$. On the derivative side of the root identity, after the usual passage from $\tilde{H}_0$ to H_0^+, this yields an integrand of $i\frac{H_0^+(\xi)}{\xi^2}\cdot(i\xi) = -\frac{H_0^+(\xi)}{\xi_+}$; this in turn leads to a Césaro divergent integral (because log-divergent) in its action on $e^{iz_0\xi}$, since we no longer have the coefficient $\frac{1}{\Gamma(0)}$ to normalise this into a δ-function. However, this is not really a problem - it merely mirrors the corresponding Césaro divergence of $\sum \frac{-1}{z_0+j}$ on the root side, for which we needed to perform the renormalisation described earlier in verifying the $\mu = 1$ root identity; a corresponding renormalisation is of course required on the derivative side.

With this in mind, we consider also the other terms. For $\mu = 1$, the term $z^{\frac{1}{2}}$ gives integrand contribution $-\frac{i}{2}H_0^+(\xi)$, and each term $e^{\frac{B_{2k}}{2k\cdot(2k-1)}\frac{1}{z^{2k-1}}}$ gives integrand contribution $\frac{B_{2k}}{2k\cdot(2k-1)}\cdot(-i)^{2k-1}\frac{\xi_+^{2k-2}}{(2k-2)!}(i\xi) = -i^{2k}\frac{B_{2k}}{(2k)!}\xi_+^{2k-1}$ (by 12). Thus, on the derivative side of the root identity, we get

$$
\begin{aligned}
&\frac{-1}{\Gamma(1)}\int_0^\infty\left\{-\frac{1}{\xi}-\frac{i}{2}-\sum_{k=1}^\infty i^{2k}\frac{B_{2k}}{(2k)!}\xi_+^{2k-1}\right\}\mathrm{e}^{iz_0\xi}\mathrm{d}\xi \\
&= \quad i\int_0^\infty\left\{\frac{1}{i\xi}+\frac{1}{2}+\sum_{k=1}^\infty\frac{B_{2k}}{(2k)!}(i\xi)^{2k-1}\right\}\mathrm{e}^{iz_0\xi}\mathrm{d}\xi \quad .
\end{aligned}
$$

But, recalling the generating function for the Bernoulli numbers (namely that $\frac{1}{\mathrm{e}^x-1} = \frac{1}{x}-\frac{1}{2}+\sum_{k=1}^\infty\frac{B_{2k}}{(2k)!}x^{2k-1}$) we see that this becomes

$$
\begin{aligned}
i\int_0^\infty\left\{\frac{1}{\mathrm{e}^{i\xi}-1}+1\right\}\mathrm{e}^{iz_0\xi}\mathrm{d}\xi &= i\int_0^\infty\left\{-\sum_{j=1}^\infty\mathrm{e}^{ij\xi}\right\}\mathrm{e}^{iz_0\xi}\mathrm{d}\xi \\
&= -i\int_0^\infty\left\{\sum_{j=1}^\infty\mathrm{e}^{i(z_0+j)\xi}\right\}\mathrm{d}\xi \\
&= -i\sum_{j=1}^\infty\left(\frac{-1}{i(z_0+j)}\right) = \sum_{j=1}^\infty\frac{1}{z_0+j}
\end{aligned}
$$

where the last step follows from a direct Césaro computation of the integrals. Since this formally equals the root side of the $\mu = 1$ root identity we see that, (up to renormalisation on both sides), $\sqrt{2\pi}\,z^{z+\frac{1}{2}}e^{-z}\mathrm{e}^{J(z)}$ does indeed satisfy the $\mu = 1$ root identity for $\Gamma(z+1)$, without requiring any further correction terms of the form e^{az}.

For $\mu = 2$ and higher, no renormalisation issues arise. At $\mu = 2$, on the derivative side, $z^z e^{-z}$ contributes $\frac{-1}{z_0}$; $z^{\frac{1}{2}}$ contributes $\frac{1}{2}\frac{1}{z_0^2}$; e^{-z} makes no contribution; and $e^{\frac{B_{2k}}{2k\cdot(2k-1)}\frac{1}{z^{2k-1}}}$ contributes $-B_{2k}\frac{1}{z_0^{2k+1}}$ after $2k$-fold Césaro integration by parts. We thus end up with $d_\Gamma(z_0,2) = \frac{1}{z_0}+\frac{1}{2z_0^2}-\sum_{m=1}^\infty\frac{B_{2m}}{z_0^{2m+1}}$ and this is a well-known expression for $-\frac{d^2}{dz^2}\left(\ln(\Gamma(z+1))\right)|_{z=z_0}$. Since we have already verified that this equals the root side of the generalised root identity for $\Gamma(z+1)$ (i.e. that $\Gamma(z+1)$ does satisfy the $\mu = 2$ root identity) so it follows that $\sqrt{2\pi}\, z^{z+\frac{1}{2}} e^{-z} e^{J(z)}$ again satisfies the $\mu = 2$ root identity without need of any further correction factor e^{az^2}. Since this is true for arbitrary z_0 without need of renormalisation, so it follows also for all $\mu \in \mathbb{Z}_{\geq 3}$ and thus Stirling's theorem is finally proven.

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