Root Identities II: Root identities for ζ - Part A

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Abstract

What has roots as nobody sees,
Is taller than trees,
Up up it goes ...

We consider the generalised root identities for the Riemann zeta function, $\zeta(s)$. We derive an explicit formula for the derivative side, $d_\zeta(s_0, \mu)$, and demonstrate its analytic continuation to all $\mu \in \mathbb{C}$, at least for $Re(s_0) > 1$. Using this we conduct initial numerical testing of the generalised root identities for ζ in the region $Re(\mu) > 1$, where the root side, $r_\zeta(s_0, \mu)$, is classically convergent. We then extend the root side to $Re(\mu) \leq 1$, using generalised geometric Césaro methods together with the Riemann-von Mangoldt formula. We adapt this generalised Césaro approach into an asymptotic expression for $r_\zeta(s_0, \mu)$ and use this to extend our numerical testing to $Re(\mu) \leq 1$, $\mu \notin \mathbb{Z}$. We thereby demonstrate that the generalised root identities do appear to be satisfied for $\mu \in \mathbb{R} \setminus \mathbb{Z}$. We then commence detailed consideration of the remaining cases of $\mu \in \mathbb{Z}_{\leq 0}$. We calculate $r_\zeta(s_0, \mu)$ for $\mu = 0, -1$ and -2 within our generalised Césaro framework, conditional on assuming the Riemann hypothesis (RH). These calculations show the criticality of the geometric location of the non-trivial roots in order to render the calculations convergent at these μ-values. For $\mu = 0$ and $\mu = -1$ the calculations show that the generalised root identities continue to be satisfied there, while for $\mu = -2$ we find that the requirement for the root identities to hold there would impose a first new condition on ζ, namely a new integral result regarding the behaviour of its argument function, $S(T)$. Once we prove rigorously in our next paper that the generalised root identities do indeed hold for ζ, this will become a general result regarding $S(T)$, conditional on RH.

1 Introduction

In [VII] we developed the generalised root identities:

$$\frac{-1}{\Gamma(\mu)} \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{\mu} (\ln(f(z)))\Big|_{z=z_0} = \mathrm{e}^{i\pi\mu} \sum_{\{z_0 - roots\, r_i\}} \frac{M_i}{(z_0 - r_i)^{\mu}}\,, \quad \mu \in \mathbb{C} \quad . \qquad (1)$$

These should hold for general $\mu \in \mathbb{C}$ and $s_0 \in \mathbb{C}$, for a representative function, f, within each equivalence class of functions sharing the same generalised root set and differing by a nowhere-zero entire factor. This is the first in a set of three papers exploring these identities in detail for the Riemann zeta function.

In [VII] we showed that ζ does satisfy the generalised root identities for all $\mu \in \mathbb{Z}_{>1}$, and also for $\mu = 1$ modulo a simple obstruction which is easily removed. For these cases, the root identities are equivalent in content to Hadamard's theorem for the function $\xi(s)$. However, we did not consider these identities more widely for $\mu \in \mathbb{C} \setminus \mathbb{Z}_{\geq 1}$ - including for $\mu \in \mathbb{Z}_{\leq 0}$, which are the cases we have seen are particularly valuable for investigating the behaviour of a function near ∞ and the asymptotic distribution of its roots. In this paper we begin this extension, while retaining for now the same practical and computational focus we adopted in [VII].

In section 2 we take up the derivative side, $d_\zeta(s_0, \mu)$. Using the Euler product formula for $\zeta(s)$, we derive an expression for $d_\zeta(s_0, \mu)$ which is readily seen to extend to an analytic function in $\mu \in \mathbb{C}$, initially for $Re(s_0) > 1$ but then for general s_0. From this, the fact that $d_\zeta(s_0, \mu)$ is identically zero when $\mu \in \mathbb{Z}_{\leq 0}$ follows trivially.

In section 3 we then take up the root side, $r_\zeta(s_0, \mu)$ and the question of whether ζ satisfies the generalised root identities for general $\mu, s_0 \in \mathbb{C}$, not just μ a positive integer.

For $Re(\mu) > 1$ the sums defining $r_\zeta(s_0, \mu)$ are all classically convergent and so, in subsection 3.1 we begin our evaluation of whether ζ continues to satisfy the generalised root identities for $\mu \notin \mathbb{Z}$ by direct *numerical* testing for suitably chosen values of μ and s_0. Initially we take both μ and s_0 in $\mathbb{R}_{>1}$ but if the root identities are satisfied there, it would follow directly by uniqueness of analytic continuation that they continue to hold, first for general $\mu \in \mathbb{C}$ and thence also for general $s_0 \in \mathbb{C}$. Based on testing using just the first 10,000 primes (on the derivative side) and the first 2,000,000 non-trivial roots (on the root side), we give strong evidence that ζ does indeed continue to satisfy the generalised root identities for all $\mu \in \mathbb{C} \setminus \mathbb{Z}$.

For $Re(\mu) \leq 1$, $\mu \neq 1$, however, the sums defining $r_\zeta(s_0, \mu)$ are classically divergent. Rather than have the continued validity of the generalised root identities hold purely as an abstract consequence of the uniqueness of analytic continuation, it is thus interesting - and for purposes of subsequent calculations which lead to new results, essential - to show how the values of $r_\zeta(s_0, \mu)$ for such μ can be calculated within a generalised geometric Césaro framework. We thus seek to demonstrate the continued validity of the root identities explicitly by numerical testing for $Re(\mu) \leq 1$, $\mu \notin \mathbb{Z}$.

To this end, in subsection 3.2, the contributions of the pole of ζ at $s = 1$ and of its trivial roots are readily deduced and extended from $Re(\mu) > 1$ to arbitrary $\mu \in \mathbb{C}$. In subsection 3.3 we then invoke the Riemann von-Mangoldt formula to attack the contribution from the non-trivial roots. Assuming RH, this allows us to develop, for $Re(s_0) > 1$, an asymptotic formula for $r_\zeta(s_0, \mu)$, analogous to the similar formulae for $r_\Gamma(z_0, \mu)$ which we derived in [VII] based on the Euler-McLaurin formula.

As promised, for non-integer μ this suffices to allow us, in subsection 3.4, to extend our numerical testing of whether ζ satisfies the generalised root identities from the half-plane $Re(\mu) > 1$, first to $0 < Re(\mu) \le 1$; then to $-1 < Re(\mu) \le 0$ and so on. We give the results of such testing, together with brief comments on the numerical issues involved.

This just leaves the question of whether ζ also satisfies the generalised root identities at the points $\mu \in \mathbb{Z}_{\le 0}$, which are of particular utility. We take up this question in section 4.

On the root side these are the μ-values where generalised Césaro summation becomes delicate. This is because the p-sum functions for the non-trivial roots, $\sum_{0<Im(\rho_i)<T} \frac{M_i}{(s_0-\rho_i)^\mu}$ and $\sum_{-\tilde{T}<Im(\rho_i)<0} \frac{M_i}{(s_0-\rho_i)^\mu}$, acquire pure log-divergences in T and $\tilde{T}$ at such values, and we know that it is such pure log-divergences which are problematic within a generalised Césaro framework and which portend the development of singularities in μ.

In subsection 4.1 we undertake the calculation of $r_\zeta(s_0,\mu)$ first for $\mu = 0$, then for $\mu = -1$ and finally for $\mu = -2$. In line with the theory developed in [I]-[III], we find that it is *critical* in these cases that our generalised Césaro calculations respect the geometric location of the non-trivial roots - namely, since we are assuming RH, that they all lie on the critical line $Re(s) = \frac{1}{2}$. Doing so is essential in generating certain extra terms within these calculations of $r_\zeta(s_0,0)$, $r_\zeta(s_0,-1)$ and $r_\zeta(s_0,-2)$, without which the generalised root identities would fail at these points.

Taking due care with these geometric considerations, however, we find that under RH, ζ does in fact satisfy the generalised root identities at $\mu = 0$ and $\mu = -1$. And for $\mu = -2$ we derive a new integral identity for the argument of the zeta function, $S(T)$, as a necessary and sufficient condition for the generalised root identities also to be satisfied there.[1]

In section 5 we conclude the paper with a brief discussion uniting our numerical testing in section 3 with the calculations just performed at $\mu = 0, -1$ and -2; and explaining the meaning of the offsetting log-divergences which arise in our generalised Césaro calculations for $r_\zeta(s_0,\mu)$ at such points $\mu \in \mathbb{Z}_{\le 0}$.

These reflections cast light on how the non-trivial roots above the real axis in fact lead to simple poles in $r_\zeta(s_0,\mu)$ at $\mu \in \mathbb{Z}_{\le 0}$, but how these are in turn cancelled by offsetting poles arising from the conjugate non-trivial roots below the real axis. These observations will be relevant in a future paper when we extend our calculations to $\mu = -3, -4, \ldots$ and thereby derive a *family* of new results regarding $S(T)$, modulo RH.

They also go hand in hand with a number of additional observations which we make here regarding how the geometry and asymptotic distribution of non-trivial roots of ζ is precisely designed to balance the contributions from its pole and its trivial roots in such a way as to guarantee that the generalised root identities are satisfied at all $\mu \in \mathbb{C} \setminus \{1\}$.

[1] In our next paper we will prove rigorously that ζ satisfies the generalised root identities for general μ, including for $\mu = -2$, and this condition will then immediately become a new result for $S(T)$ conditional on RH

2 The derivative side of the root identities for ζ

For ζ, it turns out that there is no need to invoke the full Fourier definition from [VII] in order to calculate the derivative side, $d_\zeta(s_0, \mu)$. All that is required is the property of $\left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^\mu$, introduced in [VII], that:

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^\mu (a^z)\Big|_{z=z_0} = a^{z_0}(\ln a)^\mu . \tag{2}$$

Recall the Euler product formula for ζ, namely that

$$\zeta(s) = \prod_{p\,prime} (1 - p^{-s})^{-1} \tag{3}$$

which is convergent for $Re(s) > 1$. It follows that

$$\begin{aligned} \ln(\zeta(s)) &= -\sum_{p\,prime} \ln(1 - p^{-s}) \\ &= \sum_{p\,prime} \left\{ p^{-s} + \frac{1}{2} p^{-2s} + \frac{1}{3} p^{-3s} + \dots \right\} \end{aligned} \tag{4}$$

and so, in light of property 2, we have that for $Re(s_0) > 1$ the derivative side of the root identity for ζ at μ is given by

$$d_\zeta(s_0, \mu) = -\frac{\mathrm{e}^{i\pi\mu}}{\Gamma(\mu)} \sum_{p\,prime} (\ln p)^\mu \left\{ \begin{array}{c} p^{-s_0} + 2^{\mu-1} p^{-2s_0} + 3^{\mu-1} p^{-3s_0} \\ +4^{\mu-1} p^{-4s_0} + \dots \end{array} \right\} . \tag{5}$$

This expression is convergent for arbitrary μ when $Re(s_0) > 1$ and gives an analytic expression for the derivative side of the root identities for ζ for arbitrary $\mu \in \mathbb{C}$ and $Re(s_0) > 1$.

Since, for arbitrary $Re(s_0) > 1$, the sum in equation 5 converges to some finite value as $\mu \to n$ for any $n \in \mathbb{Z}_{\leq 0}$, and since Γ has simple poles at all the non-positive integers, so a corollary of the result in equation 5 is the following:

Lemma 1: *When $\mu \in \mathbb{Z}_{\leq 0}$ the derivative sides of the generalised root identities for ζ are all identically zero as functions of s_0; that is*

$$d_\zeta(s_0, \mu) = 0 \qquad \text{for all } s_0 \text{ and for all } \mu = 0, -1, -2, \dots \tag{6}$$

If ζ satisfies the generalised root identities, we must therefore have also that

$$r_\zeta(s_0, \mu) = 0 \qquad \text{for all } s_0 \text{ and for all } \mu = 0, -1, -2, \dots \quad . \tag{7}$$

3 The root side of the root identities for ζ and testing of whether ζ satisfies them for $\mu \notin \mathbb{Z}$

On the root side, we write $r_\zeta(s_0, \mu)$ as the sum of three separate contributions from T (trivial roots), P (simple pole) and NT (non-trivial roots):

$$r_\zeta(s_0, \mu) = r_T(s_0, \mu) + r_P(s_0, \mu) + r_{NT}(s_0, \mu) \tag{8}$$

where

$$r_T(s_0, \mu) = e^{i\pi\mu} \sum_{\{s_0 - T\}} \frac{1}{(s_0 - r_i)^\mu} , \tag{9}$$

$$r_P(s_0, \mu) = -e^{i\pi\mu} \frac{1}{(s_0 - 1)^\mu} , \text{ and} \tag{10}$$

$$r_{NT}(s_0, \mu) = e^{i\pi\mu} \sum_{\{s_0 - NT\}} \frac{1}{(s_0 - \rho_i)^\mu} . \tag{11}$$

When $Re(\mu) > 1$, these are all classically convergent for generic $s_0 \in \mathbb{C}$ and so may be evaluated to any required degree of accuracy by including sufficiently many roots, both trivial and non-trivial. Here, by generic s_0 we mean s_0 not on either $(-\infty, 1)$, nor on any of the rays running from a non-trivial root $\rho_i = \beta_i + i\gamma_i$ horizontally to the left towards $-\infty$; these being branch cuts of terms in the expressions above. Since it is all that is required for our arguments, and simultaneously avoids all of these branch-cut issues, from now on we shall always restrict consideration to s_0 in the half-plane $Re(s_0) > 1$.

3.1 The case of $Re(\mu) > 1$

For such s_0 we may then readily test numerically whether ζ satisfies the generalised root identities for arbitrary $Re(\mu) > 1$ (not just for $\mu \in \mathbb{Z}_{\geq 1}$ as assessed in [VII]). We do so by simply evaluating the expressions 5 and 9-11 to the required degree of accuracy and checking whether we do indeed have

$$d_\zeta(s_0, \mu) = r_\zeta(s_0, \mu) . \tag{12}$$

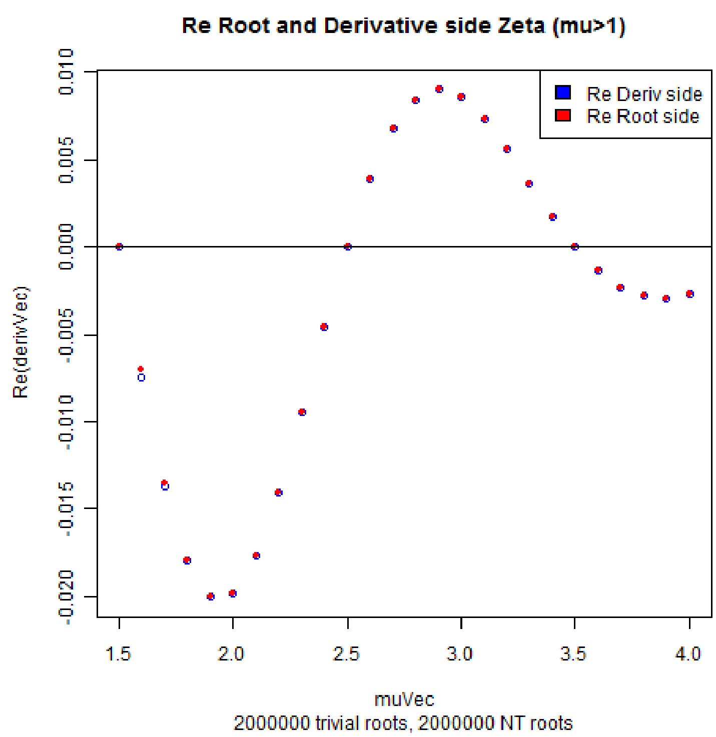
For the derivative side this requires including sufficiently many primes p (the larger s_0 is, the fewer primes required to reach a given accuracy). For the root side it requires including sufficiently many trivial roots for r_T and sufficiently many non-trivial roots for r_{NT} (the more positive $Re(\mu) > 1$ is, the fewer such roots required are for a given level of accuracy).

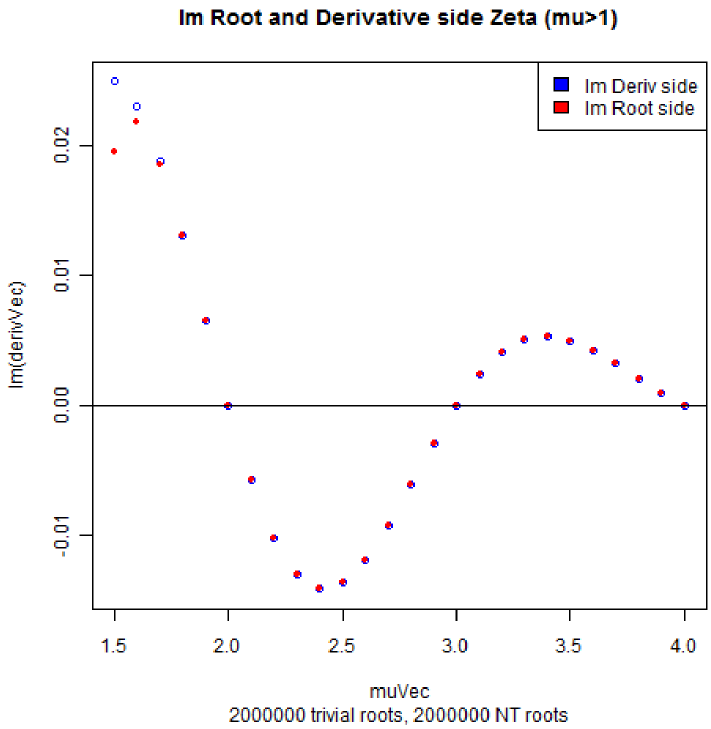
In order to implement such numerical tests a list of primes and a list of non-trivial roots must be used. Such lists can readily be obtained. For example, we shall use a list of the first $2,001,052$ NT zeros downloaded as a text file from [8], and in fact files listing up to $1,000,000,000$ NT zeros are easily accessible.

Using these we may then readily encode checks for random values of $Re(s_0) > 1$ and $Re(\mu) > 1$ to see whether ζ does indeed seem to satisfy the generalised root identities 12 for arbitrary s_0, μ in this region. We will present test results with s_0 and μ both real for simplicity, but the various sums involved in fact converge faster if either or both acquire an imaginary part.

Dana Pascovici's R-script, available with this paper, contains such code. It currently uses the first $10,000$ primes (for d_ζ) and the first $2,000,000$ trivial and NT zeros (for r_ζ), but it is readily adaptable to use more of either if desired, and thereby allows rapid, systematic checking.

On the basis of this we confirm that ζ does indeed appear to satisfy the generalised root identities for arbitrary $Re(s_0) > 1$ when $Re(\mu) > 1$, albeit that, for obvious convergence reasons, we cannot extend the testing down to μ-values too close to 1 without increasing the number of roots used in our numerical approximations for $r_\zeta(s_0, \mu)$. The following figures, for example, illustrate the agreement of both real and imaginary parts of $d_\zeta(s_0, \mu)$ and $r_\zeta(s_0, \mu)$ for $s_0 = 5.1238$ for a range of μ-values greater than 1.5.





3.2 The case of $Re(\mu) \leq 1$: Evaluation of r_P and r_T

When $Re(\mu) < 1$ the evaluation of both r_T and r_{NT} on the root side requires Césaro methods, since the sums in equations 9 and 11 are no longer classically convergent. In this section we perform the generalised Césaro evaluation of r_P and r_T. In the next we perform it for r_{NT}. In all cases we extend far enough to cover the case of $-1 < Re(\mu) \leq 1$, but it is clear how the extension of the analysis to $Re(\mu) \leq -1$ would proceed.

Taking the components $r_P(s_0, \mu)$ and $r_T(s_0, \mu)$ of the root side in turn:

Part (a) - [$r_P(s_0, \mu)$]: This is clearly given by

$$r_P(s_0, \mu) = -\mathrm{e}^{i\pi\mu} \frac{1}{(s_0 - 1)^{\mu}} \tag{13}$$

for any s_0 and μ.

Part (b) - [$r_T(s_0, \mu)$]: For $r_T(s_0, \mu)$ we use the Euler-McLaurin sum formula for the partial sums s_T. Since the shifted roots occur at $s_0 + 2j$, $j \in \mathbb{Z}_{\geq 1}$, we let $z = s_0 + 2k + \alpha$, with $0 \leq \alpha < 2$, and we write

$$r_T(s_0, \mu) = \mathrm{e}^{i\pi\mu} \cdot \underset{z \to \infty}{Clim}\, s_T(s_0, \mu; z) \tag{14}$$

where

$$s_T(s_0,\mu;z) = \sum_{j=1}^{k} \frac{1}{(s_0+2j)^\mu} \quad . \tag{15}$$

Now, by Euler-McLaurin,

$$\sum_{j=1}^{k} \frac{1}{(s_0+2j)^\mu} = \left\{ \begin{array}{c} \frac{1}{2}\frac{(s_0+2k)^{1-\mu}}{1-\mu} + C(s_0,\mu) + \frac{1}{2}(s_0+2k)^{-\mu} \\ -\frac{1}{6}\mu(s_0+2k)^{-\mu-1} \\ +\frac{\mu(\mu+1)(\mu+2)}{90}(s_0+2k)^{-\mu-3} - \ldots \end{array} \right\} \tag{16}$$

and so we can obtain the Césaro limit in equation 14 by the usual generalised Césaro-style calculations on terms of the form $(s_0+2k+\alpha)^{-\mu-l}$ (see [I]-[III]).

For example, for the case $0 < Re(\mu) \leq 1$ ($\mu \neq 1$), we have

$$\begin{aligned} (s_0+2k+\alpha)^{1-\mu} &= (s_0+2k)^{1-\mu}\left\{1+(1-\mu)\frac{\alpha}{(s_0+2k)}+\ldots\right\} \\ &= (s_0+2k)^{1-\mu} + o(1) \end{aligned} \tag{17}$$

and thus since pure powers z^ρ ($\rho \neq 0$) have generalised Césaro limit 0, so

$$r_T(s_0,\mu) = \mathrm{e}^{i\pi\mu} \mathop{lim}_{k\to\infty} \left\{ \sum_{j=1}^{k} \frac{1}{(s_0+2j)^\mu} - \frac{1}{2}\frac{(s_0+2k)^{1-\mu}}{1-\mu} \right\} \tag{18}$$

where this is now a classical limit.

Similar, but more involved, calculations can be performed on the lower order terms to extend equation 18 to a formula applicable for $-1 < Re(\mu) \leq 0$ ($\mu \neq 0$), then $-2 < Re(\mu) \leq -1$ ($\mu \neq -1$) and so on. With each step, an additional Césaro averaging is now required. We shall only consider the next case of $-1 < Re(\mu) \leq 0$ ($\mu \neq 0$) here. In the case the required strong Césaro asymptotic relationship we need to use is that

$$P\left[\frac{1}{2}(s_0+2k)^{-\mu}(\alpha-1)\right] = o(1) \tag{19}$$

and in this case the formula becomes

$$r_T(s_0,\mu) = \mathrm{e}^{i\pi\mu} \mathop{lim}_{k\to\infty} \left\{ \sum_{j=1}^{k} \frac{1}{(s_0+2j)^\mu} - \frac{1}{2}\frac{(s_0+2k)^{1-\mu}}{1-\mu} - \frac{1}{2}(s_0+2k)^{-\mu} \right\} \quad . \tag{20}$$

3.3 The case of $Re(\mu) \leq 1$: Evaluation of r_{NT}

We now turn to evaluating $r_{NT}(s_0,\mu)$ for $Re(\mu) \leq 1$. Here, rather than use the Euler-McLaurin formula, we instead use the Riemann-von Mangoldt formula for the counting function $N(T)$ giving the number of non-trivial roots, ρ_i, with $0 < Im(\rho_i) < T$.

Write $NT = NT_+ \cup NT_-$ where NT_+ refers to the subset of non-trivial zeros with positive imaginary part and NT_- to the conjugate subset with negative imaginary part. We work initially on NT_+ and then simply state the corresponding formulae for NT_-, which are derived in analogous fashion.

For a root $\rho_i = \beta_i + i\gamma_i \in NT_+$ we have, on writing $\beta_i = \frac{1}{2} + \epsilon_i$, that

$$\begin{aligned}
\frac{1}{(s_0 - (\beta_i + i\gamma_i))^\mu} &= (s_0 - \frac{1}{2} - \epsilon_i - i\gamma_i)^{-\mu} \\
&= \mathrm{e}^{i\frac{\pi}{2}\mu}\gamma_i^{-\mu}\left(1 + i\frac{(s_0 - \frac{1}{2} - \epsilon_i)}{\gamma_i}\right)^{-\mu} \\
&= \mathrm{e}^{i\frac{\pi}{2}\mu}\gamma_i^{-\mu}\left\{\begin{array}{c} 1 - i\mu\frac{(s_0-\frac{1}{2}-\epsilon_i)}{\gamma_i} \\ -\frac{\mu(\mu+1)}{2!}\frac{(s_0-\frac{1}{2}-\epsilon_i)^2}{\gamma_i^2} \\ +i\frac{\mu(\mu+1)(\mu+2)}{3!}\frac{(s_0-\frac{1}{2}-\epsilon_i)^3}{\gamma_i^3} + \ldots \end{array}\right\}
\end{aligned} \tag{21}$$

Therefore, bearing in mind that any roots off the critical line occur in symmetric pairs either side of it so that terms with odd powers in ϵ_i may be ignored, we have

$$\sum_{\{s_0 - NT_+\}} \frac{M_i}{(s_0 - \rho_i)^\mu} = \mathrm{e}^{i\frac{\pi}{2}\mu}\left\{\begin{array}{c} \sum_{\{s_0-NT_+\}} M_i\gamma_i^{-\mu} \\ -i\mu(s_0 - \frac{1}{2})\sum_{\{s_0-NT_+\}} M_i\gamma_i^{-\mu-1} \\ -\frac{\mu(\mu+1)(s_0-\frac{1}{2})^2}{2!}\sum_{\{s_0-NT_+\}} M_i\gamma_i^{-\mu-2} \\ -\frac{\mu(\mu+1)}{2!}\sum_{\{s_0-NT_+\}} M_i\gamma_i^{-\mu-2}\epsilon_i^2 \\ +i\frac{\mu(\mu+1)(\mu+2)(s_0-\frac{1}{2})^3}{3!}\sum_{\{s_0-NT_+\}} M_i\gamma_i^{-\mu-3} \\ +i\frac{\mu(\mu+1)(\mu+2)(s_0-\frac{1}{2})}{2}\sum_{\{s_0-NT_+\}} M_i\gamma_i^{-\mu-3}\epsilon_i^2 + \ldots \end{array}\right\} \tag{22}$$

We next need to express the partial sums for these series in terms of the parameter T. As mentioned, we do this using the Riemann-von Mangoldt formula which states (see [11]) that

$$N(T) = \check{N}(T) + S(T) + \frac{1}{\pi}\delta(T) \quad . \tag{23}$$

Here

$$\check{N}(T) = \frac{T}{2\pi}\ln(\frac{T}{2\pi}) - \frac{T}{2\pi} + \frac{7}{8} \tag{24}$$

and $S(T)$ is the argument of the zeta function.[2] As for $\delta(T)$, it is given by the following explicit formula (in terms of the saw-tooth function $\overset{\vee}{q}_0(u)$ defined previously in [III]) and consequent estimate:

$$\delta(T) = \frac{T}{4}\ln\left(1+\frac{1}{4T^2}\right)+\frac{1}{4}\tan^{-1}\left(\frac{1}{2T}\right)+\frac{T}{2}\int_0^\infty \frac{\overset{\vee}{q}_0(u)}{(u+\frac{1}{4})^2+(\frac{T}{2})^2}\mathrm{d}u \tag{25}$$

$$= O\left(\frac{1}{T}\right) \quad \text{as } T\to\infty \quad . \tag{26}$$

Once we have used $N(T)$ to derive expressions for the p-sums in equation 22 in terms of T, we will then re-express these in terms of the appropriate *geometric* variable z and try to evaluate their generalised Césaro limits in the usual way (see [I]-[III]), by removing eigenfunctions and generalised eigenfunctions (z^ρ, $z^\rho \ln z$ etc) and averaging the remainder (in T).

Note, however, that in using $N(T)$ for these derivations, only the sums in equation 22 which have no explicit ϵ_i-dependence will be able to be handled in general. This is because we certainly have no understanding of the functional dependence of any $\epsilon_i(T)$ on T.

Since the sums involving ϵ_i remain classically convergent for $-1 < Re(\mu) \leq 1$, and since we have for now restricted our attention to such μ, we could safely ignore this observation for purposes of our numerical testing of the generalised root identities in this region in the rest of this section.

However, since it immediately becomes relevant when we progress further left in the μ-plane or when we seek exact rather than merely approximate numerical results, we shall, for the rest of this paper and for the next two in this series on ζ, remove this concern by henceforward assuming the Riemann hypothesis, namely:

Assumption [RH]: *All the non-trivial roots, ρ_i, in the critical strip lie on the critical line, i.e. $Re(\rho_i) = \frac{1}{2}$.*

This makes $\epsilon_i = 0$ for all i and so, in the spirit of Henry II (pers. comm.), rids us of these troublesome terms, while making the results we derive in the rest of this series all conditional on RH. We return to discuss this dependency at the end of the last paper (part C) in the series.

For now, however, suppose $-1 < Re(\mu) \leq 1$, ($\mu \neq 1$). Then in equation 22 it is only the sums $\sum_{\{s_0-NT_+\}} M_i\gamma_i^{-\mu}$ and $\sum_{\{s_0-NT_+\}} M_i\gamma_i^{-\mu-1}$ which may be divergent - the former for all such μ, the latter only when $Re(\mu) < 0$.

Let us take these two sums in turn.

[2] Morally speaking $S(t)$ is the argument of ζ on the critical line, i.e. $S(t) = \frac{1}{\pi}Im(\ln(\zeta(\frac{1}{2}+it)))$, but care needs to be taken regarding conventions and handling of the non-trivial roots (see [9])

First sum [$\sum_{\{s_0-NT_+\}} M_i\gamma_i^{-\mu}$]: For this the p-sum up to imaginary part T is

$$\int_0^T t^{-\mu} dN(t) \quad = \quad T^{-\mu}N(T) + \mu \int_0^T t^{-\mu-1} N(t)\, \mathrm{d}t \tag{27}$$

$$= \left\{ \begin{array}{c} T^{-\mu}\check{N}(T) + \mu \int_0^T t^{-\mu-1}\check{N}(t)\, \mathrm{d}t \\ \\ +T^{-\mu}S(T) + \mu \int_0^T t^{-\mu-1}S(t)\, \mathrm{d}t \\ \\ +\frac{1}{\pi}T^{-\mu}\delta(T) + \frac{1}{\pi}\mu \int_0^T t^{-\mu-1}\delta(t)\, \mathrm{d}t \end{array} \right\} . \tag{28}$$

We shall consider the contributions arising from $\check{N}(t)$, $S(t)$ and $\delta(t)$ within this equation separately in turn, but first we make one key comment.

Comment: Since the first non-trivial root arises at approximately $t = 14.13473$, $N(t)$ is identically zero in a neighbourhood of $t = 0$, with all its three components being continuous near 0 (and $\check{N}(0) = \frac{7}{8}$, $\frac{1}{\pi}\delta(0) = \frac{1}{8}$ and $S(0) = -1$). As such, it is only divergences in the terms in equation 28 which arise from behaviour as $T \to \infty$ that need to be identified and handled within our geometric Césaro framework in order to arrive at a formula for the NT-root side contribution which is capable of numerical evaluation when $-1 < Re(\mu) \leq 1$. We can safely discard any evaluations (in particular any divergences) within our integrals at the lower limit 0, since any such contributions will cancel when the pieces for $\check{N}(t)$, $S(t)$ and $\delta(t)$ are combined. To this end, we shall henceforth simply write $\int^T$ rather than $\int_0^T$ and only evaluate our integrals at their upper limit, T.[3]

Returning now to equation 28 and taking its pieces in turn:

Component (i) [The $\delta(t)$ pieces]: Since $\delta(T) = O(\frac{1}{T})$, we have that for $Re(\mu) > -1$, $T^{-\mu}\,\delta(T) \to 0$ as $T \to \infty$ and the integrand $t^{-\mu-1}\,\delta(t)$ is classically integrable as $t \to \infty$. As such, for $-1 < Re(\mu) \leq 1$, the terms involving $\delta(t)$ introduce no divergences requiring removal by generalised Césaro means into equation 28.

Component (ii) [The $S(t)$ pieces]: For the $S(t)$ pieces in equation 28, we use the following result, which combines theorems 9.4 and 9.9A from [10]:

Result 1: *We have that*

$$S(T) = O(logT) \qquad \text{as} \quad T \to \infty\,. \tag{29}$$

[3] If this reasoning is thought too loose by a fastidious reader, such reader could alternatively think of $\check{N}(t)$, $S(t)$ and $\delta(t)$ replaced wlog throughout by truncated versions of themselves which are set identically zero to the left of some arbitrary fixed truncation point a with $0 < a < 14.13473$. Alternatively, the reader could more fully embrace the Césaro framework and use it also to handle any divergences which arise in our integrals near 0, since all such divergences will in fact be Césaro eigenfunctions or generalised eigenfunctions with generalised Césaro limit 0. Personally, I am happy with any of these approaches.

Moreover, if we define $S_1(T)$ *by* $S_1(T) := \int_0^T S(t)\,dt$*, then we also have that*

$$S_1(T) = O(logT) \qquad \text{as} \quad T \to \infty\,. \tag{30}$$

Ignoring lower-limit integrability issues as discussed, let us use this to examine first the term $T^{-\mu}S(T)$. By equation 29, if $0 < Re(\mu) \leq 1$ then $T^{-\mu}S(T)$ already classically converges to 0 as $T \to \infty$. And if $-1 < Re(\mu) \leq 0$ then we have that

$$\begin{aligned} P\left[t^{-\mu}S(t)\right](T) &= \frac{1}{T}\int_0^T t^{-\mu}S(t)\,dt \\ &= T^{-\mu-1}S_1(T) + \frac{\mu}{T}\int_0^T t^{-\mu-1}S_1(t)\,dt + o(1)\,. \end{aligned}$$

Now, by equation 30 we have that $T^{-\mu-1}S_1(T) = o(1)$ (since $Re(\mu) > -1$). Since the Césaro operator is regular, so also $\frac{\mu}{T}\int_0^T t^{-\mu-1}S_1(t)\,dt \to 0$ as $T \to \infty$. Thus, when $-1 < Re(\mu) \leq 0$, $T^{-\mu}S(T)$ is strongly Césaro convergent to 0 via a single application of P. Overall, there is no divergence arising from the term $T^{-\mu}S(T)$ when $0 < Re(\mu) \leq 1$ and only an oscillatory divergence which is handled by a single application of P when $-1 < Re(\mu) \leq 0$.

As for the integral involving $S(t)$ in equation 28, in the same way we have that

$$\int^T t^{-\mu-1}S(t)\,\mathrm{d}t = T^{-\mu-1}S_1(T) + (\mu+1)\int^T t^{-\mu-2}S_1(t)\,dt + o(1) \quad .$$

Now, whenever $-1 < Re(\mu) \leq 1$, equation 30 implies that the term $T^{-\mu-1}S_1(T)$ is classically convergent to 0, and also that $\int^\infty t^{-\mu-2}S_1(t)\,dt$ is classically integrable so that $\int^T t^{-\mu-2}S_1(t)\,dt$ introduces no divergences in T as $T \to \infty$.

Thus, overall, the two terms involving $S(t)$ in equation 28 introduce no divergences as $T \to \infty$ when $0 < Re(\mu) \leq 1$; and when $-1 < Re(\mu) \leq 0$ they only introduce such mild oscillatory divergences as are rendered classically convergent to 0 by a single application of the Césaro averaging operator, P.

Component (iii) [The $\check{N}(t)$ pieces]: The only terms where there are eigenfunction power-divergences that require removal within the Césaro framework are therefore those involving $\check{N}(T)$. From equation 24, on writing $u = \frac{T}{2\pi}$, using integration by parts and changing variables to $v = \frac{t}{2\pi}$, we find these are

$$T^{-\mu}\check{N}(T) + \mu\int^T t^{-\mu-1}\check{N}(t)\,\mathrm{d}t = (2\pi)^{-\mu}\left\{\begin{array}{c}\left[u^{1-\mu}\ln u - u^{1-\mu} + \frac{7}{8}u^{-\mu}\right] \\ +\mu\int^u \left[\begin{array}{c} v^{-\mu}\ln v - v^{-\mu} \\ +\frac{7}{8}v^{-\mu-1}\end{array}\right]\mathrm{d}v\end{array}\right\}$$

$$\begin{aligned} &= (2\pi)^{-\mu}\left\{\begin{array}{c}\frac{1}{1-\mu}\left(\frac{T}{2\pi}\right)^{1-\mu}\ln\left(\frac{T}{2\pi}\right) \\ -\frac{1}{(1-\mu)^2}\left(\frac{T}{2\pi}\right)^{1-\mu}\end{array}\right\} \\ &+C + o(1) \quad . \end{aligned} \tag{31}$$

Second sum $[\sum_{\{s_0-NT_+\}} M_i\gamma_i^{-\mu-1}]$: Similarly, in this case the p-sum is given by

$$\int^T t^{-\mu-1}dN(t) = T^{-\mu-1}N(T) + (\mu+1)\int^T t^{-\mu-2}N(t)\,\mathrm{d}t \tag{32}$$

and for $-1 < Re(\mu) \leq 1$, the terms arising from the $\delta(t)$-pieces and $S(t)$-pieces are now always classically convergent. The only divergent terms are again the power and power-log divergences coming from the $\check{N}(T)$ terms, namely

$$T^{-\mu-1}\check{N}(T) + (\mu+1)\int^T t^{-\mu-2}\check{N}(t)\,\mathrm{d}t \;\; = (2\pi)^{-\mu-1} \left\{ \begin{array}{c} -\frac{1}{\mu}\left(\frac{T}{2\pi}\right)^{-\mu}\ln\left(\frac{T}{2\pi}\right) \\ -\frac{1}{\mu^2}\left(\frac{T}{2\pi}\right)^{-\mu} \end{array} \right\} + C + o(1) \quad . \tag{33}$$

Combining sums: It follows from combining these calculations in equation 22 that, when $-1 < Re(\mu) \leq 1$, $\mu \notin \{0,1\}$, the divergent part of the p-sum for $\sum_{\{s_0-NT_+\}} \frac{M_i}{(s_0-\rho_i)^\mu}$ which cannot be tamed simply by applying pure powers of P is given by $divgt_{NT_+}(s_0,\mu;T)$ where

$$\begin{aligned} divgt_{NT_+}(s_0,\mu;T) \;\; &= \;\; \mathrm{e}^{i\frac{\pi}{2}\mu}(2\pi)^{-\mu} \left\{ \begin{array}{c} \frac{1}{1-\mu}\left(\frac{T}{2\pi}\right)^{1-\mu}\ln\left(\frac{T}{2\pi}\right) \\ -\frac{1}{(1-\mu)^2}\left(\frac{T}{2\pi}\right)^{1-\mu} \end{array} \right\} \\ &\quad + i\mathrm{e}^{i\frac{\pi}{2}\mu}(2\pi)^{-\mu-1}(s_0-\frac{1}{2}) \left\{ \begin{array}{c} \left(\frac{T}{2\pi}\right)^{-\mu}\ln\left(\frac{T}{2\pi}\right) \\ +\frac{1}{\mu}\left(\frac{T}{2\pi}\right)^{-\mu} \end{array} \right\} \end{aligned} \tag{34}$$

Now, in the geometric Césaro framework, we need to re-express these divergent pieces in terms, not of T, but of the relevant *geometric* variable z, which respects the *location* of our root-summands. Only then are we authorised (see [I]-[III]) to consider $\underset{z\to\infty}{Clim}$ (rather than simply $\underset{T\to\infty}{Clim}$).

Since we are assuming RH, so that $\frac{1}{2}+it$ traces out the positive imaginary half of the critical line, so our summands occur along the ray $s_0 - \frac{1}{2} - it$ and we take $z = s_0 - \frac{1}{2} - iT$. Then

$$\frac{T}{2\pi} = i\cdot\frac{z}{2\pi} - i\cdot\frac{(s_0-\frac{1}{2})}{2\pi} = i\cdot\frac{z}{2\pi}\cdot\left\{1 - \frac{(s_0-\frac{1}{2})}{z}\right\} \quad . \tag{35}$$

We substitute this in equation 34, apply the binomial expansion for powers of $\frac{T}{2\pi}$, and note also that as $z\to\infty$

$$\begin{aligned} \ln\frac{T}{2\pi} \;\; &= \;\; i\frac{\pi}{2} + \ln\frac{z}{2\pi} + \ln\left(1 - \frac{(s_0-\frac{1}{2})}{z}\right) \\ &= \;\; i\frac{\pi}{2} + \ln\frac{z}{2\pi} - \frac{(s_0-\frac{1}{2})}{z} - \frac{1}{2}\frac{(s_0-\frac{1}{2})^2}{z^2} - \ldots \quad . \end{aligned} \tag{36}$$

We find that, for $-1 < Re(\mu) \leq 1$, $\mu \notin \{0,1\}$, $divgt_{NT_+}(s_0,\mu;T)$ can be re-expressed as

$$divgt_{NT_+}(s_0,\mu;z) = \left\{ \begin{array}{c} i\,\frac{(2\pi)^{-\mu}}{1-\mu}\left(\frac{z}{2\pi}\right)^{1-\mu}\ln\left(\frac{z}{2\pi}\right) \\ -(2\pi)^{-\mu}\left\{\frac{\frac{\pi}{2}}{1-\mu}+\frac{i}{(1-\mu)^2}\right\}\left(\frac{z}{2\pi}\right)^{1-\mu} \\ +(2\pi)^{-\mu-1}\frac{i}{\mu}(s_0-\frac{1}{2})\left(\frac{z}{2\pi}\right)^{-\mu} \end{array} \right\} + o(1) \quad (37)$$

But since $\mu \neq 0, 1$, all these terms have generalised Césaro limit 0 so that

$$\underset{z\to\infty}{Clim}\, divgt_{NT_+}(s_0,\mu;z) = 0 \qquad \text{when } \mu \in (-1,1)\setminus\{0\} \quad . \quad (38)$$

It follows, for $-1 < Re(\mu) \leq 1$, $\mu \notin \{0,1\}$, that the generalised geometric Césaro sum $\sum_{\{s_0-NT_+\}}\frac{M_i}{(s_0-\rho_i)^\mu}$ is given by taking the p-sum, removing $divgt_{NT_+}(s_0,\mu;T)$ and then either taking the classical limit of the residual sum (if $Re(\mu) > 0$, $\mu \neq 1$), or else taking this classical limit after applying one power of the Césaro averaging operator, P, to it (if $-1 < Re(\mu) \leq 0$, $\mu \neq 0$).

That is,

$$\sum_{\{s_0-NT_+\}}\frac{M_i}{(s_0-\rho_i)^\mu} = \underset{T\to\infty}{Clim}\left\{\sum_{Im(\rho_i)<T}\frac{M_i}{(s_0-\rho_i)^\mu} - divgt_{NT_+}(s_0,\mu;T)\right\} \quad (39)$$

where $divgt_{NT_+}(s_0,\mu;T)$ is as per equation 34 and this is in fact a classical limit if $Re(\mu) > 0$, $\mu \neq 1$; and requires one application of P (averaging in T) if $-1 < Re(\mu) \leq 0$, $\mu \neq 0$.

The corresponding calculation for NT_-: In similar fashion, writing $\rho_i = (\frac{1}{2}+\epsilon_i) - i\gamma_i$ for the non-trivial roots below the real axis, we have for NT_- that

$$\sum_{\{s_0-NT_-\}}\frac{M_i}{(s_0-\rho_i)^\mu} = \mathrm{e}^{-i\frac{\pi}{2}\mu}\left\{ \begin{array}{c} \sum_{\{s_0-NT_-\}} M_i\gamma_i^{-\mu} \\ +i\mu(s_0-\frac{1}{2})\sum_{\{s_0-NT_-\}} M_i\gamma_i^{-\mu-1} \\ -\frac{\mu(\mu+1)}{2!}(s_0-\frac{1}{2})^2\sum_{\{s_0-NT_-\}} M_i\gamma_i^{-\mu-2} \\ -\frac{\mu(\mu+1)}{2!}\sum_{\{s_0-NT_-\}} M_i\gamma_i^{-\mu-2}\epsilon_i^2 \\ -i\frac{\mu(\mu+1)(\mu+2)}{3!}(s_0-\frac{1}{2})^3\sum_{\{s_0-NT_-\}} M_i\gamma_i^{-\mu-3} \\ -i\frac{\mu(\mu+1)(\mu+2)}{2}(s_0-\frac{1}{2})\sum_{\{s_0-NT_-\}} M_i\gamma_i^{-\mu-3}\epsilon_i^2 + \dots \end{array} \right\} \quad (40)$$

Assuming RH and applying identical reasoning, for the p-sum $\sum_{-\tilde{T}<Im(\rho_i)<0}\frac{M_i}{(s_0-\rho_i)^\mu}$, and for $-1 < Re(\mu) \leq 1$, $\mu \notin \{0,1\}$, we have that the pieces arising from $\delta(t)$

are again all classically convergent. Likewise the pieces arising from $S(t)$ are either classically convergent (if $Re(\mu) > 0,\ \mu \neq 1$) or else require just one Césaro-averaging (if $-1 < Re(\mu) \leq 0,\ \mu \neq 0$). And the only power-divergences requiring removal within the Césaro framework arise from the $\check{N}(t)$-pieces and are given by

$$divgt_{NT_-}(s_0,\mu;\tilde{T}) = \mathrm{e}^{-i\frac{\pi}{2}\mu}(2\pi)^{-\mu}\left\{\begin{array}{c} \frac{1}{1-\mu}\left(\frac{\tilde{T}}{2\pi}\right)^{1-\mu}\ln\left(\frac{\tilde{T}}{2\pi}\right) \\ -\frac{1}{(1-\mu)^2}\left(\frac{\tilde{T}}{2\pi}\right)^{1-\mu} \end{array}\right\} \\ -i\mathrm{e}^{-i\frac{\pi}{2}\mu}(2\pi)^{-\mu-1}(s_0-\frac{1}{2})\left\{\begin{array}{c} \left(\frac{\tilde{T}}{2\pi}\right)^{-\mu}\ln\left(\frac{\tilde{T}}{2\pi}\right) \\ +\frac{1}{\mu}\left(\frac{\tilde{T}}{2\pi}\right)^{-\mu} \end{array}\right\} \tag{41}$$

For $\mu \neq 0, 1$ this can be re-expressed in terms of the geometric variable $\tilde{z} = (s_0 - \frac{1}{2}) + i\tilde{T}$ as

$$divgt_{NT_-}(s_0,\mu;\tilde{z}) = \left\{\begin{array}{c} -i\,\frac{(2\pi)^{-\mu}}{1-\mu}\left(\frac{\tilde{z}}{2\pi}\right)^{1-\mu}\ln\left(\frac{\tilde{z}}{2\pi}\right) \\ \\ -(2\pi)^{-\mu}\left\{\frac{\frac{\pi}{2}}{1-\mu}-\frac{i}{(1-\mu)^2}\right\}\left(\frac{\tilde{z}}{2\pi}\right)^{1-\mu} \\ \\ -(2\pi)^{-\mu-1}\frac{i}{\mu}(s_0-\frac{1}{2})\left(\frac{\tilde{z}}{2\pi}\right)^{-\mu} \end{array}\right\} + o(1) \tag{42}$$

which has generalised Césaro limit 0, i.e.

$$\underset{\tilde{z}\to\infty}{Clim}\, divgt_{NT_-}(s_0,\mu;\tilde{z}) = 0 \quad . \tag{43}$$

Thus, for $-1 < Re(\mu) \leq 1,\ \mu \notin \{0,1\}$ we have the Césaro sum

$$\sum_{\{s_0-NT_-\}} \frac{M_i}{(s_0-\rho_i)^\mu} = \underset{\tilde{T}\to\infty}{Clim}\left\{\sum_{Im(\rho_i)>-\tilde{T}} \frac{M_i}{(s_0-\rho_i)^\mu} - divgt_{NT_-}(s_0,\mu;\tilde{T})\right\} \tag{44}$$

where $divgt_{NT_-}(s_0,\mu;\tilde{T})$ is as given in equation 41 and again this is a classical limit if $Re(\mu) > 0,\ \mu \neq 1$ and requires one application of the averaging operator, $P_{\tilde{T}}$, if $-1 < Re(\mu) \leq 0,\ \mu \neq 0$.

If we write $\tilde{r}_{NT_+}(s_0,\mu)$ and $\tilde{r}_{NT_-}(s_0,\mu)$ for the expressions on the right in equations 39 and 44 we thus have

$$r_{NT}(s_0,\mu) = \mathrm{e}^{i\pi\mu}\sum_{\{s_0-NT\}} \frac{M_i}{(s_0-\rho_i)^\mu} = r_{NT_+}(s_0,\mu) + r_{NT_-}(s_0,\mu) \tag{45}$$

where

$$r_{NT_+}(s_0,\mu) = \mathrm{e}^{i\pi\mu}\tilde{r}_{NT_+}(s_0,\mu) \qquad \text{and} \qquad r_{NT_-}(s_0,\mu) = \mathrm{e}^{i\pi\mu}\tilde{r}_{NT_-}(s_0,\mu) \quad . \tag{46}$$

The expressions in equations 39 and 44 allow numerical computation of $r_{NT}(s_0, \mu)$ also for $-1 < Re(\mu) \leq 1$, $\mu \notin \{0, 1\}$, via equations 45 and 46.

3.4 Numerical tests of the generalised root identities for ζ for $-1 < Re(\mu) \leq 1$, $\mu \notin \{0, 1\}$

Based on the results for $r_P(s_0, \mu)$, $r_T(s_0, \mu)$ and $r_{NT}(s_0, \mu)$ derived in subsections 3.2 and 3.3, we have implemented code for numerical evaluation of $r_\zeta(s_0, \mu)$ and thus for extending our earlier testing of the generalised root identities for ζ in subsection 3.1 to the region $-1 < Re(\mu) \leq 1$, $\mu \notin \{0, 1\}$. The following general observations apply to these efforts:

Observations: (a) As in subsection 3.1, we will only present here testing results for μ real, i.e. $\mu \in (-1, 1)$, $\mu \neq 0$. All the calculations in fact converge more rapidly if μ acquires an imaginary part, however, so this is purely for simplicity, and such checks for non-real μ can easily be performed if desired.

(b) With only $2,000,000$ NT_+ roots, convergence is too slow to obtain reasonable approximate values on the root side when μ approaches 0 from *above*. This is for the same reason that we couldn't obtain reasonable approximations even in the convergent region ($\mu > 1$) when μ approached 1 from above.

When μ is just above an integer like 0, the residual piece, even after throwing away the divergent terms, converges too slowly (like $T^{-\mu}$ or $T^{-\mu}S(T)$ when $\mu > 0$ small) to be able to get reasonable convergence with only $2,000,000$ NT_+ roots.

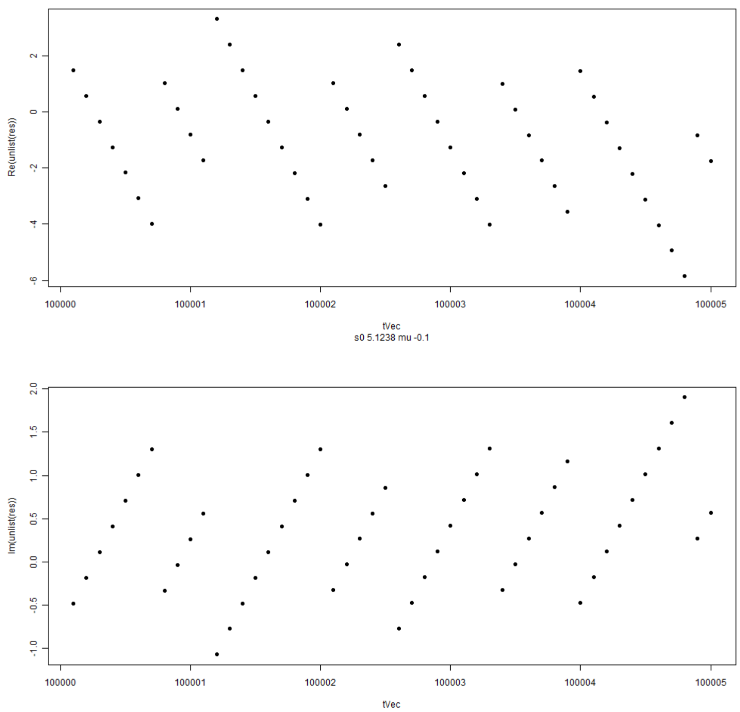
However, when μ approaches an integer like 0 from *below*, one *can* get accurate numerical results, since on throwing away the extra divergent terms which appear when μ passes through the integer value, the remaining residual term decays much more rapidly (like $T^{-1-\mu}$ or $T^{-1-\mu}S(T)$ when $\mu < 0$ small).

Thus for our numerical tests in this section we consider only μ-regions just to the *left* of $\mu = 1$ and 0. This also helps get accurate evaluations of the trivial-root contributions, $r_T(s_0, \mu)$, with our choice of $2,000,000$ trivial roots.

(c) As noted, when $0 < \mu < 1$ we only need to remove divergent pieces and then take classical limits (and in doing this we only subtract those terms in the expressions 20, 34 and 41 which are themselves divergent when $0 < \mu < 1$).

When $-1 < \mu < 0$, however, we need to remove the divergent pieces in equations 34 and 41 from the NT-root contributions and then apply the Césaro averaging operator (in T or $\tilde{T}$) once to the resulting partial-sum function on the critical line. This final Césaro averaging is needed in order to handle the step-jumps on crossing roots, which are now bigger than 1 and which grow. That is to say, it is needed to handle the oscillatory contributions from the $S(T)$-components of these expressions.

Now, when μ is small negative and approaching 0 (the testing region we are focussing on) these step-discontinuities approach height 1 and the resulting partial-sum functions become very close to piecewise linear functions with negative slope between root-steps. This can be seen clearly, for example, in the following picture, which shows the partial-sum function for $\tilde{r}_{NT_+}(s_0, \mu)$ when $s_0 = 5.1238$ and $\mu = -0.1$ on the interval $100,000 < T < 100,005$:



We see that when $\mu < 0$ is small, we can use the trapezoidal rule between each pair of successive NT-roots to perform the required Césaro averaging approximately, and hence numerically evaluate the contributions from $r_{NT_+}(s_0, \mu)$ and $r_{NT_-}(s_0, \mu)$. This is what we have implemented in our code. It allows us to verify whether ζ does indeed continue to satisfy the generalised root identities also when $\mu < 0$ is small, i.e. to test whether whether the root-side of these identities, $r_\zeta(s_0, \mu)$, does tend to 0 as $\mu \to 0^-$, as the derivative side $d_\zeta(s_0, \mu)$ does (per Lemma 1).

(d) With the observations from (b) and (c) in mind, in implementing our numerical root-side calculations for ζ we have:

(i) implemented this within R-script for $0 < \mu < 1$;[4] however

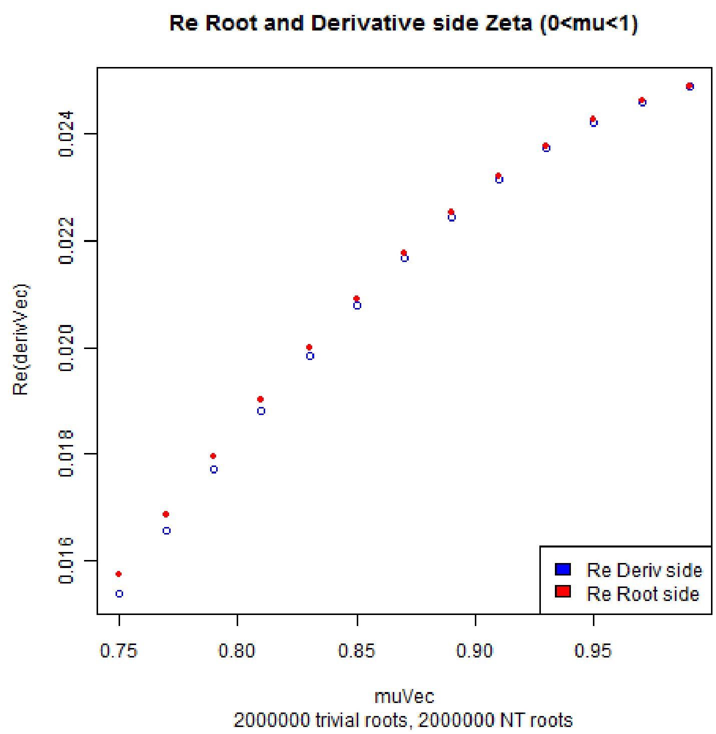
(ii) when $-1 < \mu < 0$ we have implemented the full calculations involving

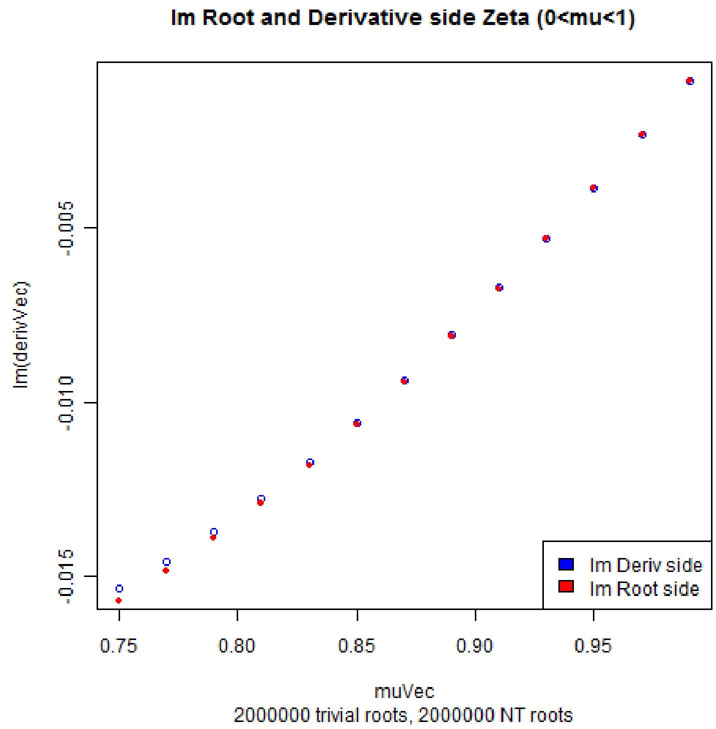
[4]We also used R-script to generate pictures of the partial-sum function in T when $-1 < \mu < 0$, as in the figure above

Césaro averaging in VBA code in XL2007. This could be adapted to the more powerful framework of R, but VBA is sufficient for our numerical experiments in this paper.

Both the Excel file ("RootIdentitiesZeta mu -1To0 TestsB.xlsm") and the zeta R-script, along with their required source files, are made available with this paper. Thea reader may readily examine the code involved, replicate the claimed results and conduct further independent tests if desired.

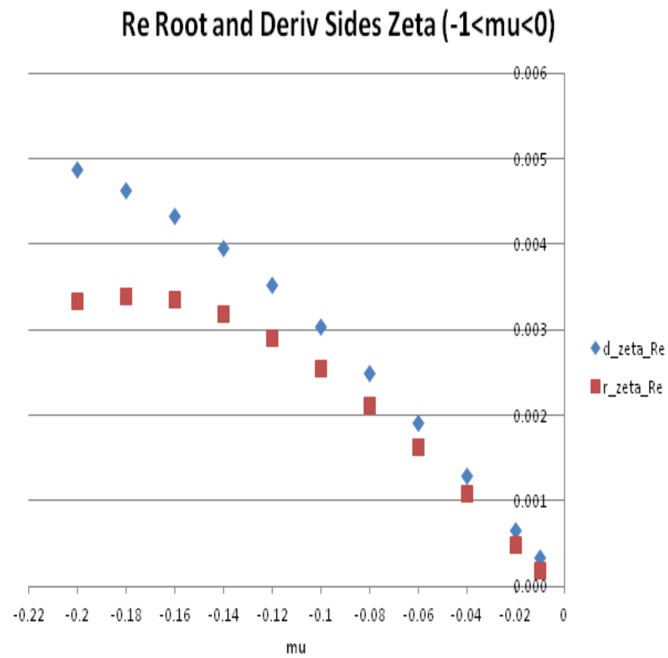
Results: **(a)** When $0 < \mu < 1$ we get close agreement between $d_\zeta(s_0, \mu)$ and $r_\zeta(s_0, \mu)$ for $\mu > 0.75$. For $\mu < 0.75$, as noted, $2,000,000$ NT roots is insufficient to gain a good approximation to the NT-root contributions to $r_\zeta(s_0, \mu)$, with the situation growing worse as $\mu \to 0^+$. The following figures (for $s_0 = 5.1238$) show how the convergence on the root side improves appreciably as $\mu \to 1^-$ and agreement between $d_\zeta(s_0, \mu)$ and $r_\zeta(s_0, \mu)$ thus becomes progressively better:

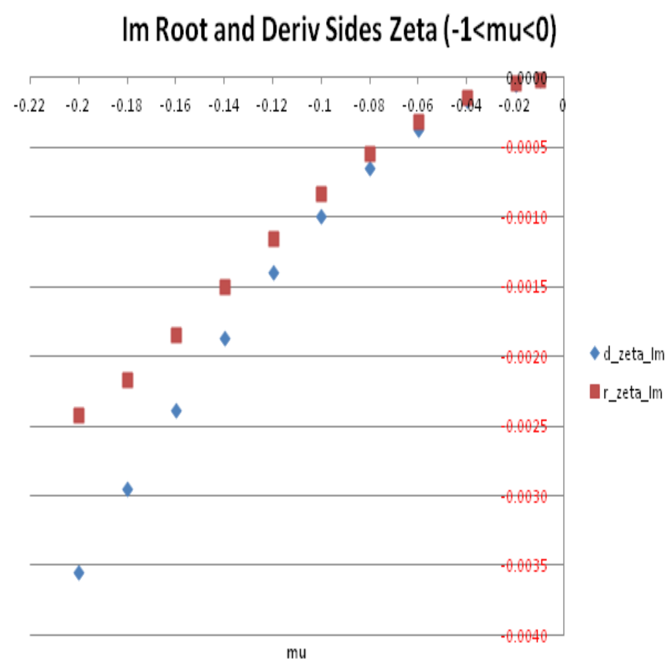




We believe that these pictures provide convincing evidence that ζ does indeed satisfy the generalised root identities also for $0 < \mu < 1$. We could test more accurately, and over a wider μ-range, by using more than $2,000,000$ trivial and NT-roots.

(b) When $-1 < \mu < 0$ we likewise get reasonably close agreement between $d_\zeta(s_0, \mu)$ and $r_\zeta(s_0, \mu)$ for $\mu > -0.2$ after Césaro averaging; but $2,000,000$ NT-roots is insufficient to obtain reasonable root side values when $\mu < -0.2$:


Re Root and Deriv Sides Zeta (-1<mu<0)
0.006
0.005
0.004
0.003
0.002
0.001
0.000
-0.22 -0.2 -0.18 -0.16 -0.14 -0.12 -0.1 -0.08 -0.06 -0.04 -0.02 0
mu
d_zeta_Re
r_zeta_Re


Im Root and Deriv Sides Zeta (-1<mu<0)
-0.22 -0.2 -0.18 -0.16 -0.14 -0.12 -0.1 -0.08 -0.06 -0.04 -0.02 0
0.0000
-0.0005
-0.0010
-0.0015
-0.0020
-0.0025
-0.0030
-0.0035
-0.0040
mu
d_zeta_Im
r_zeta_Im

These figures are again for $s_0 = 5.1238$. Here, and in all testing described for $-1 < \mu < 0$, the approximations are obtained using $10,000$ primes on the derivative side with a truncation threshold of 10 (so that $d_\zeta(s_0,\mu)$ values should be very accurate); while on the root side we use $2,000,000$ trivial roots for r_T and an average over the next $10,000$ roots starting at the $1,000,000^{th}$ NT-root, with 5 subintervals in the trapezoidal rule and an offset of 10^{-9} (to avoid the step-jumps at roots) in the calculation of r_{NT_+} and r_{NT_-}.

Overall, bearing in mind the caveats we have discussed regarding the relatively crude nature of the numerical testing conducted here and its consequent accuracy, we believe that our additional testing of cases with $-1 < Re(\mu) \leq 1$ continues to support the belief that ζ does satisfy the generalised root identities at all $\mu \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$.

Note that, as $\mu \to 0^-$, not only does agreement between $d_\zeta(s_0,\mu)$ and $r_\zeta(s_0,\mu)$ become very close, but both quantities also approach 0.[5] This provides strong evidence that the values of $d_\zeta(s_0,\mu)$ and $r_\zeta(s_0,\mu)$ behave continuously across the integer point $\mu = 0$ - with these values approaching the expected limiting value from Lemma 1 of 0 when $\mu = 0$ - and suggesting the generalised root identities continue to be satisfied also at $\mu \in \mathbb{Z}_{\leq 0}$. We now turn to this question.

4 The generalised root identities for ζ at $\mu \in \mathbb{Z}_{\leq 0}$ - Calculations of $r_\zeta(s_0,\mu)$ at $\mu = 0, -1, -2$ and a new result

Having seen that ζ appears to satisfy the generalised root identities at all $\mu \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, and also seen some preliminary evidence that this continues to hold at $\mu \in \mathbb{Z}_{\leq 0}$, we devote this section to specifically considering the simplest three such integer-μ cases of $\mu = 0, -1$ and -2.

The cases of $\mu \in \mathbb{Z}_{\leq 0}$ are, as discussed, the most interesting cases to consider. But they are also the most delicate, from the point of view of the Césaro analysis required to calculate the root sides, $r_\zeta(s_0,\mu)$, and hence to check whether they are all identically zero for such μ, in agreement with $d_\zeta(s_0,\mu)$. This delicateness arises for two reasons.

First, because at these points the p-sum component arising from the non-trivial roots acquires log-divergences and these need to be handled carefully since, within a generalised Césaro framework, the development of such log-divergences indicates the presence of poles in μ at the μ-values in question.

Secondly, because it is at these integer values of μ that the *geometric* aspects of generalised geometric Césaro summation become *critical*.

At $\mu \notin \mathbb{Z}$ such considerations are ultimately not material, but we will see that it is only by abiding strictly by these geometric requirements (i.e. performing

[5] albeit there seems to be a threshold on the root side reflecting the limits of accuracy we can obtain using only $2,000,000$ trivial roots and an average over only $10,000$ roots starting at the $1,000,000^{th}$ NT-root in our approximate calculations

our Césaro calculations with respect to the geometric complex variables z and $\tilde{z}$, rather than the real parameters T and $\tilde{T}$) that we are able to get the correct value of zero for $r_\zeta(s_0, \mu)$ at $\mu = 0$ and -1, while at the same time satisfactorily explaining the cancellation of the poles associated with the log-divergences there.

For $\mu = -2$, we likewise need to retain this geometric perspective in our Césaro calculations. When we do so we will see that obtaining $r_\zeta(s_0, -2) = 0$ for all $Re(s_0) > 1$ imposes a further strong Césaro condition on the limiting behaviour of $S(t)$ as $t \to \infty$ and this leads to the first of a family of new results we will derive for $S(t)$ from the generalised root identities, conditional on RH.

Preliminaries: To simplify our calculations we introduce the following notational preliminaries which apply in everything that follows:

For a given T, we denote by w and $\tilde{w}$ the scaled geometric variables given by

$$w = \frac{z}{2\pi} \qquad (resp. \quad \tilde{w} = \frac{\tilde{z}}{2\pi}) \tag{47}$$

and by u and $\tilde{u}$ the corresponding scaled parameters

$$u = \frac{T}{2\pi} \qquad (resp. \quad \tilde{u} = \frac{\tilde{T}}{2\pi})\,. \tag{48}$$

Letting

$$q = \frac{(s_0 - \frac{1}{2})}{2\pi} \tag{49}$$

we then have

$$u = iw \cdot \left(1 - \frac{q}{w}\right) \qquad (resp. \quad \tilde{u} = -i\tilde{w} \cdot \left(1 - \frac{q}{\tilde{w}}\right)) \tag{50}$$

so that, for example, equation 36 becomes

$$\ln u = i\frac{\pi}{2} + \ln w - \frac{q}{w} - \frac{1}{2}\frac{q^2}{w^2} - \ldots \qquad (resp. \quad \ln \tilde{u} = -i\frac{\pi}{2} + \ln \tilde{w} - \frac{q}{\tilde{w}} - \frac{1}{2}\frac{q^2}{\tilde{w}^2} - \ldots) \tag{51}$$

while dilation-invariance also means that $\underset{z,\tilde{z}\to\infty}{Clim}$ and $\underset{w,\tilde{w}\to\infty}{Clim}$ are interchangeable.

Since in section 3 we only considered $Re(\mu) > -1$, the cases of $\mu = -1$ and $\mu = -2$ will also require extension of some of our earlier working. To begin with, we will need to extend the formulae for $r_T(s_0, \mu)$ we derived in subsection 3.2.

As for $r_{NT}(s_0, \mu)$, things are a little easier in one sense in that at $\mu = 0, -1$ or -2 we can take the lower limit as 0 in our integral calculations for r_{NT_+} and r_{NT_-} without any issue concerning divergences at this lower limit - which simplifies things. But we will need to derive additional results regarding the $\delta(t)$-pieces and the $\check{N}(t)$-pieces, and we will also need to take care in setting up how we handle the $S(t)$-pieces in terms of Césaro-adapted scales (see [III]). Let us now start, however, with the simplest case of $\mu = 0$.

4.1 The case of $\mu = 0$

Here, in equation 13 we have $r_P(s_0, 0) = -1$, and by equation 20 $r_T(s_0, 0) = -\frac{1}{2}s_0 - \frac{1}{2}$, so it is only the NT-root contribution that requires further analysis.

When $\mu = 0$ we get in equations 22 and 40 that the partial sum for $r_{NT_+}(s_0, 0) + r_{NT_-}(s_0, 0)$, which we shall denote by $s_{+,-}(s_0, 0; T, \tilde{T})$, is given by

$$\begin{aligned} s_{+,-}(s_0, 0; T, \tilde{T}) &= \sum_{\{\gamma_i < T\}} M_i + \sum_{\{\tilde{\gamma}_i < \tilde{T}\}} M_i \\ &= N(T) + N(\tilde{T}) \overset{C}{\sim} \check{N}(T) + \check{N}(\tilde{T}) \quad . \end{aligned} \tag{52}$$

Here we have ignored the $\delta(T)$ terms since they are $O(\frac{1}{T})$, and invoked Result 1 from section 3.3 to ignore the Césaro limit of the terms $S(T)$ and $S(\tilde{T})$, since $P[S](T) = \frac{1}{T}S_1(T) = O\left(\frac{\ln T}{T}\right) = o(1)$.

But by equations 50 and 51 it follows that, in terms of w and $\tilde{w}$, this partial sum is Césaro asymptotic to

$$\begin{aligned} s_{+,-}(s_0, 0; T, \tilde{T}) &\overset{C}{\sim} \left\{ u \ln u - u + \frac{7}{8} \right\} + \left\{ \tilde{u} \ln \tilde{u} - \tilde{u} + \frac{7}{8} \right\} \\ &= \left\{ \begin{array}{c} iw \ln w - (\frac{\pi}{2} + i)w \\ -i\tilde{w} \ln \tilde{w} - (\frac{\pi}{2} - i)\tilde{w} + iq \ln\left(\frac{\tilde{w}}{w}\right) \end{array} \right\} + (\frac{7}{4} + \pi q) \end{aligned} \tag{53}$$

$$\begin{aligned} &\overset{C}{\to} \frac{7}{4} + \pi q + iq \cdot \underset{w, \tilde{w} \to \infty}{Clim} \ln\left(\frac{\tilde{w}}{w}\right) \\ &= \frac{1}{2}s_0 + \frac{3}{2} + iq \cdot \underset{w, \tilde{w} \to \infty}{Clim} \ln\left(\frac{\tilde{w}}{w}\right) \end{aligned} \tag{54}$$

and it follows that

$$r_{NT_+}(s_0, 0) + r_{NT_-}(s_0, 0) = \frac{1}{2}s_0 + \frac{3}{2} + iq \cdot \underset{w, \tilde{w} \to \infty}{Clim} \ln\left(\frac{\tilde{w}}{w}\right) \quad .$$

Combining with our results for $r_P(s_0, 0)$ and $r_T(s_0, 0)$ we see that

$$r_\zeta(s_0, 0) = iq \cdot \underset{w, \tilde{w} \to \infty}{Clim} \ln\left(\frac{\tilde{w}}{w}\right) . \tag{55}$$

Thus, in order to have $r_\zeta(s_0, 0) = 0$, so that ζ satisfies the generalised root identities at $\mu = 0$, it is necessary and sufficient to have that $\underset{w, \tilde{w} \to \infty}{Clim} \ln\left(\frac{\tilde{w}}{w}\right) = 0$.

But this is in fact precisely the requirement to ensure the *analyticity* of our extension by generalised Césaro methods. In other words, despite the fact that (per [I]) neither $\ln \tilde{w}$ nor $\ln w$ have generalised Césaro limits independently, we have the following result:

Result 2: *In order for our extension via generalised Césaro methods to be analytic, we must augment these methods with the stipulation that*

$$\underset{w,\tilde{w}\to\infty}{Clim} \ln\left(\frac{\tilde{w}}{w}\right) = \underset{z,\tilde{z}\to\infty}{Clim} \ln\left(\frac{\tilde{z}}{z}\right) = 0\,. \tag{56}$$

Proof: The final expression in set brackets in equation 53 is identical to what we obtain by taking the limit as $\mu \to 0$ in the sum of the two divergent pieces, $divgt_{NT_+}(s_0, \mu; z)$ and $divgt_{NT_-}(s_0, \mu; \tilde{z})$, given in equations 37 and 42. This is because we have $w^{-\mu} = 1 - \mu \ln w + O(\mu^2)$ (resp. $\tilde{w}^{-\mu} = 1 - \mu \ln \tilde{w} + O(\mu^2)$) as $\mu \to 0$, which allows us to combine the final terms in each equation and deduce that $iq\left(\frac{1}{\mu}w^{-\mu} - \frac{1}{\mu}\tilde{w}^{-\mu}\right) \to iq \cdot \ln\left(\frac{\tilde{w}}{w}\right)$ in the limit.

But then, since the Césaro limit of the sum of these divergent pieces is identically zero for all $\mu \neq 0$, the only way for $r_\zeta(s_0, \mu)$ to remain analytic as μ crosses 0 is for the Césaro limit of this limiting expression to also be zero. This in turn is obviously equivalent to having $\underset{w,\tilde{w}\to\infty}{Clim} \ln\left(\frac{\tilde{w}}{w}\right) = 0$ - since the other terms in w and $\tilde{w}$ are all Césaro eigenfunctions or generalised eigenfunctions with eigenvalue $\frac{1}{2}$ and thus automatically have generalised Césaro limit 0 - and so the result follows.

With this extra stipulation, it follows that we do indeed have

Lemma 2:

$$r_\zeta(s_0, 0) = 0 \qquad \text{for arbitrary } Re(s_0) > 1 \tag{57}$$

and so ζ does also satisfy the generalised root identities at $\mu = 0$, at least for all s_0 in the specified half-plane.

Before moving on to the case of $\mu = -1$, however, we first make a couple of additional observations.

Comments: **(i)** The derivation above of Result 2 is one which we shall have recourse to and explain more thoroughly in the next paper in this series, when we provide a rigorous *proof* that ζ satisfies the generalised root identities at all $\mu \in \mathbb{C} \setminus \{1\}$.

It is, however, a little bloodless, so it is worth explaining more concretely *why* $iq \cdot \ln\left(\frac{\tilde{w}}{w}\right) = iq \cdot \ln \tilde{w} - iq \cdot \ln w$ needs to be taken as converging to 0 in a generalised Césaro sense - especially given that $\ln \tilde{w}$ and $\ln w$ are Césaro eigenfunctions with eigenvalue 1 and thus not independently able to be ascribed generalised Césaro limits.

In fact, if it were on its own, the presence of the term $-iq \cdot \ln w$ in the p-sum *at* $\mu = 0$ *would* indicate the presence of a pole, $iq \cdot \frac{1}{\mu}$, for μ near 0. This term arises from the contribution of the non-trivial roots *above* the real axis - specifically, from the final term in equation 37 for the p-sum of $r_{NT_+}(s_0, \mu)$ where, since $w^{-\mu} \to 1$ as $\mu \to 0$, we have that

$$(2\pi)^{-\mu-1}\frac{i}{\mu}(s_0 - \frac{1}{2})\left(\frac{z}{2\pi}\right)^{-\mu} \approx iq \cdot \frac{1}{\mu}$$

for μ small.

If this were the only log-divergence, then the p-sum at $\mu = 0$ would not be Césaro-summable and a pole in μ would exist. However, the term $iq \cdot \ln \tilde{w}$ provides an offsetting log-divergence. It arises in the same way, but with opposite overall sign, from the contribution of the non-trivial roots *below* the real axis. Specifically, it arises from the last term in equation 42, which has

$$-(2\pi)^{-\mu-1}\frac{i}{\mu}(s_0 - \frac{1}{2})\left(\frac{\tilde{z}}{2\pi}\right)^{-\mu} \approx -iq \cdot \frac{1}{\mu}$$

for μ near 0.

The non-trivial roots below the real axis thus end up providing an exactly offsetting pole in μ near 0! This leaves a finite value, namely $r_\zeta(s_0, 0) = 0$, as part of the well-defined, analytic behaviour of $r_\zeta(s_0, \mu)$ as μ crosses 0.

We can see this very clearly numerically by considering the following tables. These show in detail the breakdown of the real and imaginary parts of $d_\zeta(s_0, \mu)$ and $r_\zeta(s_0, \mu)$ when μ is very small, namely $\mu = -0.001$, $\mu = -0.0001$ and $\mu = -0.00001$:

μ	$Re(d_\zeta)$	$Re(r_\zeta)$	$Re(r_T)$	$Re(r_P)$	$Re(r_{NT_+})$	$Re(r_{NT_-})$
-0.001	0.00003300	-0.00009450	-3.064361	-1.001413	4.348981	-0.283302
-0.0001	0.00000330	-0.00012270	-3.062147	-1.000142	4.343429	-0.281263
-0.00001	0.00000033	-0.00012547	-3.061925	-1.000014	4.342870	-0.281056

μ	$Im(d_\zeta)$	$Im(r_\zeta)$	$Im(r_T)$	$Im(r_P)$	$Im(r_{NT_+})$	$Im(r_{NT_-})$
-0.001	-0.0000001037	0.0000002974	0.009627	0.003146	737.242	-737.255
-0.0001	-0.0000000010	0.0000000385	0.000962	0.000314	7,360.36	-7,360.36
-0.00001	-0.0000000000	0.0000000039	0.000096	.000031	73,591.4	-73,591.4

While the root-side (like the derivative side) does approach 0 as $\mu \to 0^-$, the imaginary components arising from r_{NT_+} and r_{NT_-} do not themselves become small but rather diverge as $\mu \to 0^-$. For the choice of $s_0 = 5.1238$ shown, these divergences more and more closely approach $\pm iq \cdot \frac{1}{\mu}$ as μ goes from -0.001 to -0.00001, exactly as promised. But these pure-imaginary poles arising from NT_+ and NT_- cancel each other out, leaving analytic behaviour in $r_{NT} = r_{NT_+} + r_{NT_-}$, and hence also in r_ζ, as μ crosses 0.

On the real side, note that the real components of $r_{NT_+} + r_{NT_-}$ also more and more closely approach the expected value of $\frac{1}{2}s_0 + \frac{3}{2}$ as μ approaches 0 (just as r_T and r_P more and more closely approach $-\frac{1}{2}s_0 - \frac{1}{2}$ and -1 respectively).

(ii) The *geometric* aspects of geometric generalised Césaro convergence are *critical* in deriving the correct value of 0 for $r_\zeta(s_0, 0)$, rather than have $\mu = 0$ being an anomaly or removable singularity of $r_\zeta(s_0, \mu)$ (see [I] for examples of such anomalies).[6]

[6]For example in the extension of ζ itself under a discrete Césaro framework.

If we applied our generalised Césaro framework to u and $\tilde{u}$ in the working leading to equation 53, rather than to w and $\tilde{w}$, then the constant term in equation 54 would have been simply $\frac{7}{4}$, rather than $\frac{7}{4}+\pi q$. This alone would not have cancelled the contributions from $r_T(s_0,0)+r_P(s_0,0)$ and so would not have left us with $r_\zeta(s_0,0)=0$ as desired.

Indeed, without the extra πq term - which arises from the extra $\pm\frac{i\pi}{2}$ term in going from $\ln u$ to $\ln w$ (resp. from $\ln\tilde{u}$ to $\ln\tilde{w}$) - we would have no s_0-dependence at all in $r_{NT}(s_0,0)$ to cancel the s_0-dependence of the contribution from $r_T(s_0,0)$.

We thus see that respecting these geometric aspects in using generalised Césaro methodology to analytically continue $r_\zeta(s_0,\mu)$, is not only essential, but makes these Césaro methods particularly well-adapted for exploring the behaviour of the non-trivial roots of zeta, especially at $\mu\in\mathbb{Z}_{\leq 0}$.

(iii) To expand on this last point briefly, since we are adding in 1 for each generalised root, the case of $\mu=0$ considered in this subsection represents a formal (or "renormalised") *count* of the roots of ζ - overall and within each subgroup (P, T and NT). The observation in (ii) thus gives a first glimpse of a new way of understanding the non-trivial roots and their location.

Specifically, they play the role of perfectly balancing the known trivial roots of ζ and its pole, so as to leave ζ satisfying the generalised root identities at all $\mu\in\mathbb{C}\setminus\{1\}$ and general s_0.

This viewpoint should be of particular interest when $\mu\in\mathbb{Z}_{\leq 0}$ for several reasons.

First, because these are the points where the generalised Césaro methodology involved in analytically continuing $r_\zeta(s_0,\mu)$ becomes sensitive to geometric considerations.

And secondly because when $Re(\mu)<1$ the p-sums defining the components of $r_\zeta(s_0,\mu)$ become classically divergent and so become as sensitive to the location and contribution of distant roots approaching ∞ as they are to "small" roots. This was not the case for the instances of $\mu\in\mathbb{Z}_{\geq 1}$ which correspond to Hadamard's theorem for ζ.

Thus the cases of the generalised root identities for $Re(\mu)<1$, and in particular for $\mu\in\mathbb{Z}_{\leq 0}$ - where we combine this sensitivity to the asymptotic distribution of "distant" non-trivial roots with Césaro sensitivity to their geometric location - hold out hope for giving new insights into ζ and its mysterious non-trivial roots.

As a first trivial application of this way of thinking, note for example that once we know that ζ does satisfy the generalised root identity at $\mu=0$, we can immediately conclude that there must be infinitely many non-trivial roots. This fact is obviously well-known and follows at once from the Riemann-von Mangoldt formula, but is not otherwise self-evident. Why can we immediately deduce it? Because if there were only finitely many such roots, say N, then we would have $r_{NT}(s_0,0)=N$ independent of s_0. Since $r_T(s_0,0)=-\frac{1}{2}s_0-\frac{1}{2}$ is dependent on s_0, it would follow at once that we could not have $r_\zeta(s_0,0)=d_\zeta(s_0,0)=0$ for arbitrary $Re(s_0)>1$ as required, a contradiction.

4.2 The case of $\mu = -1$

We now turn to calculating $r_\zeta(s_0, -1)$ and checking the generalised root identities for the case of $\mu = -1$. As discussed, we need to start by extending the working for $Re(\mu) > -1$ in section 3 to cover this case, and also that of $\mu = -2$.

4.2.1 Extending results for $r_T(s_0, \mu)$

In order to extend from $Re(\mu) > -1$ to, say, $Re(\mu) > -3$, formula 16 needs no extension, and we still compare this with the binomial expansion of $\frac{1}{2}\frac{1}{1-\mu}z^{1-\mu}$ where $z = s_0+2k+\alpha$. However we need to push this binomial expansion further than in equation 17, namely:

$$\frac{1}{2}\frac{z^{1-\mu}}{1-\mu} = \left\{ \begin{array}{c} \frac{1}{2}\frac{(s_0+2k)^{1-\mu}}{1-\mu} + \frac{1}{2}(s_0+2k)^{-\mu}\alpha + 1 \\ -\frac{1}{4}\mu(s_0+2k)^{-\mu-1}\alpha^2 + \frac{1}{12}\mu(\mu+1)(s_0+2k)^{-\mu-2}\alpha^3 + \dots \end{array} \right\} \tag{58}$$

Then, to extend to a neighbourhood of $\mu = -1$, the strong Césaro asymptotic result that needs to be verified is that

$$P^2[\frac{1}{2}(s_0+2k)^{-\mu}(\alpha-1) - \frac{\mu}{12}(s_0+2k)^{-\mu-1}(3\alpha^2-2)] = o(1)$$

while in a neighbourhood of $\mu = -2$ it is that

$$P^3[\frac{1}{2}(s_0+2k)^{-\mu}(\alpha-1) - \frac{\mu}{12}(s_0+2k)^{-\mu-1}(3\alpha^2-2) + \frac{\mu(\mu+1)}{12}(s_0+2k)^{-\mu-2}\alpha^3] = o(1) \quad .$$

Using these it follows in general that, for $Re(\mu) > -3$, we have

$$r_T(s_0,\mu) = e^{i\pi\mu} \lim_{k\to\infty} \left\{ \begin{array}{c} \sum_{j=1}^{k} \frac{1}{(s_0+2j)^\mu} - \frac{1}{2}\frac{(s_0+2k)^{1-\mu}}{1-\mu} \\ -\frac{1}{2}(s_0+2k)^{-\mu} + \frac{\mu}{6}(s_0+2k)^{-\mu-1} \end{array} \right\} \quad . \tag{59}$$

Alternatively, to get $r_T(s_0,\mu)$ more directly, we may invoke Césaro dilation-invariance directly to place our summands, $(s_0+2j)^{-\mu}$, not at s_0+2j but rather at $\frac{s_0}{2}+j$. Then, since $(s_0+2j)^{-\mu} = 2^{-\mu}(\frac{s_0}{2}+j)^{-\mu}$, it follows from our earlier discussions of the Hurewicz zeta function $\zeta_H(z,\rho)$ (see [II]) that we have

$$r_T(s_0,\mu) = e^{i\pi\mu} \cdot 2^{-\mu} \cdot \zeta_H(\frac{s_0}{2},\mu) \quad . \tag{60}$$

Either way, it follows immediately that where for $\mu = 0$ we had $r_T(s_0, 0) = -\frac{1}{2}s_0 - \frac{1}{2}$, for $\mu = -1$ and $\mu = -2$ we have, respectively:

$$r_T(s_0,-1) = \frac{1}{4}s_0^2 + \frac{1}{2}s_0 + \frac{1}{6} \tag{61}$$

and

$$r_T(s_0,-2) = -\frac{1}{6}s_0^3 - \frac{1}{2}s_0^2 - \frac{1}{3}s_0 \quad . \tag{62}$$

Further preliminaries: We now turn to extending our results also for $r_{NT}(s_0,\mu)$, taking each component - the $\check{N}(t)$-pieces, the $\delta(t)$-pieces and the $S(t)$-pieces - in turn. To facilitate this, we need to introduce one further notational preliminary. We define a ladder of integrals of $N(t)$ by

$$N_0(T) := N(T), \quad \text{and} \quad N_i(T) := \int_0^T N_{i-1}(t)\mathrm{d}t \quad \forall i \in \mathbb{Z}_{>0} \tag{63}$$

and we similarly define corresponding ladders for each of the components $\check{N}_i(T)$, $\delta_i(T)$ and $S_i(T)$ for all $i \in \mathbb{Z}_{\geq 0}$.

4.2.2 Extending results for the $\check{N}(t)$-pieces of $r_{NT}(s_0,\mu)$

We have that $\check{N}(T) = u\ln u - u + \frac{7}{8}$ (where, as before, $u := \frac{T}{2\pi}$) and it follows from elementary integration that

$$\check{N}_1(T) = (2\pi)\cdot\left\{\frac{1}{2}u^2\ln u - \frac{3}{4}u^2 + \frac{7}{8}u\right\} \tag{64}$$

and

$$\check{N}_2(T) = (2\pi)^2\cdot\left\{\frac{1}{6}u^3\ln u - \frac{11}{36}u^3 + \frac{7}{16}u^2\right\} \quad . \tag{65}$$

From these equations we have the following results, which we will use in our calculations for the $\mu=-1$ and $\mu=-2$ cases:

$$T\check{N}(T) - \check{N}_1(T) = (2\pi)\cdot\left\{\frac{1}{2}u^2\ln u - \frac{1}{4}u^2\right\} \tag{66}$$

and

$$T^2\check{N}(T) - 2T\check{N}_1(T) + 2\check{N}_2(T) = (2\pi)^2\cdot\left\{\frac{1}{3}u^3\ln u - \frac{1}{9}u^3\right\} \quad . \tag{67}$$

4.2.3 Extending results for the $\delta(t)$-pieces of $r_{NT}(s_0,\mu)$

From equation 25 it is straightforward (see below) to see that $\delta(T)$ has an asymptotic expansion of the form

$$\delta(T) = a_1\frac{1}{T} + a_3\frac{1}{T^3} + a_5\frac{1}{T^5} + \dots \tag{68}$$

as $T\to\infty$.

In [9, sections 6.5 and 6.7] it is shown that $\check{N}(t)+\delta(t) = \frac{1}{\pi}\theta(t)+1$ where $\theta(t) = Im(log(\Gamma(\frac{1}{4}+\frac{it}{2}))) - \frac{t}{2}\ln\pi$ and, by applying Stirling's theorem giving the asymptotic behaviour of Γ, it follows that we have $a_1 = \frac{1}{48}$, $a_3 = \frac{7}{5760}$ and so on; so that in fact we have

$$\delta(T) = \frac{1}{48}\frac{1}{T} + \frac{7}{5760}\frac{1}{T^3} + \frac{31}{80640}\frac{1}{T^5} + o\left(\frac{1}{T^5}\right) . \tag{69}$$

Here, however, we will also have to calculate formulae for $\delta_1(T)$ and $\delta_2(T)$ and since this entails further calculation of successive integration constants, so equation 69 alone does not suffice.

The integrals for $\delta_1(T)$ and $\delta_2(T)$ are in fact well-defined classically and the required formulae can be computed rigorously without overstepping the bounds of existing mathematical propriety. But it turns out to be much quicker and easier to deduce them using generalised Césaro methods - which in turn provides further evidence of the utility and simplicity of such techniques. As such, as a committed evangelist for such methods and as the founding (and currently only) member of the society for their propagation, that is what we now do.

Césaro derivation of a_1: We start by deriving the fact that $a_1 = \frac{1}{48}$ by these means. As $T \to \infty$, we have that $\frac{T}{4} \ln\left(1 + \frac{1}{4T^2}\right) = \frac{1}{16}\frac{1}{T} + O(\frac{1}{T^3})$ and $\frac{1}{4}\tan^{-1}\left(\frac{1}{2T}\right) = \frac{1}{8}\frac{1}{T} + O(\frac{1}{T^3})$, and also that

$$\begin{aligned} \frac{T}{2}\int_0^\infty \frac{\overset{\vee}{q}_0(u)}{(u+\frac{1}{4})^2 + (\frac{T}{2})^2}\mathrm{d}u &= \frac{2}{T}\underset{k\to\infty}{Clim}\int_0^{k+\alpha} \frac{\overset{\vee}{q}_0(u)}{1 + 4\frac{(u+\frac{1}{4})^2}{T^2}}\mathrm{d}u \\ &= \frac{2}{T}\underset{k\to\infty}{Clim}\int_0^{k+\alpha} \overset{\vee}{q}_0(u)\mathrm{d}u + O\left(\frac{1}{T^3}\right) \quad . \end{aligned}$$

Now $\int_0^{k+\alpha} \overset{\vee}{q}_0(u)\mathrm{d}u = \frac{1}{2}\alpha^2 - \frac{1}{2}\alpha$ and since we know (see [I]) that

$$\underset{k\to\infty}{Clim}\, k^n \alpha^r = \frac{(-1)^n}{n+r+1} \tag{70}$$

so the coefficient of $\frac{1}{T}$ from this final integral term is $2 \cdot (\frac{1}{6} - \frac{1}{4}) = -\frac{1}{6}$. Thus, combining pieces, we have

$$a_1 = \frac{3}{16} - \frac{1}{6} = \frac{1}{48} \tag{71}$$

as promised.

Césaro calculation for $\delta_1(T)$: Turning next to $\delta_1(T)$, this is given by

$$\begin{aligned} \delta_1(T) &= \int_0^T \frac{t}{4}\ln\left(1 + \frac{1}{4t^2}\right)\mathrm{d}t + \frac{1}{4}\int_0^T \tan^{-1}\left(\frac{1}{2t}\right)\mathrm{d}t \\ &\quad + \frac{1}{2}\int_0^T t\int_0^\infty \frac{\overset{\vee}{q}_0(u)}{(u+\frac{1}{4})^2 + (\frac{t}{2})^2}\mathrm{d}u\mathrm{d}t \quad . \end{aligned} \tag{72}$$

Elementary integration shows that

$$\int_0^T \frac{t}{4}\ln\left(1 + \frac{1}{4t^2}\right)\mathrm{d}t = \frac{1}{8}T^2\ln\left(1 + \frac{1}{4T^2}\right) + \frac{1}{32}\ln\left(1 + 4T^2\right) \tag{73}$$

and

$$\frac{1}{4}\int_0^T \tan^{-1}\left(\frac{1}{2t}\right)\,\mathrm{d}t = \frac{1}{4}T\cdot\tan^{-1}\left(\frac{1}{2T}\right) + \frac{1}{16}\ln\left(1+4T^2\right)\,. \qquad (74)$$

As for the final integral term, reversing order of integration in the way we have seen is permissible when dealing with the Césaro scale $\overset{\vee}{q}_j(\alpha)$, we get

$$\frac{1}{2}\int_0^\infty \overset{\vee}{q}_0(u)\int_0^T \frac{t}{(u+\frac{1}{4})^2+\frac{t^2}{4}}\mathrm{d}t\,\mathrm{d}u$$

$$= \int_0^\infty \overset{\vee}{q}_0(u)\left[\ln\left((u+\frac{1}{4})^2+\frac{t^2}{4}\right)\right]_0^T \mathrm{d}u$$

$$= \left\{\begin{array}{c} \int_0^\infty \overset{\vee}{q}_0(u)\ln\left((u+\frac{1}{4})^2+\frac{T^2}{4}\right)\mathrm{d}u \\ -2\int_0^\infty \overset{\vee}{q}_0(u)\ln(u+\frac{1}{4})\,\mathrm{d}u \end{array}\right\}\,.$$

The first of the two integrals here can be expressed asymptotically in T as

$$\int_0^\infty \overset{\vee}{q}_0(u)\left\{2\ln(\frac{T}{2})+4\frac{(u+\frac{1}{4})^2}{T^2}-\ldots\right\}\mathrm{d}u$$

and a Césaro argument identical to the one just used in deriving the value of a_1 shows this is equal to

$$-\frac{1}{6}\ln(T)+\frac{1}{6}\ln 2+O(\frac{1}{T^2})\,.$$

The second of the integrals is a constant which we label A, i.e.

$$A := -\int_0^\infty \overset{\vee}{q}_0(u)\ln(u+\frac{1}{4})\,\mathrm{d}u\,. \qquad (75)$$

This can be shown to have a well-defined Césaro value $A \simeq -0.104$, but we omit details here.

Thus, overall, combining terms we get finally in equation 72 that

$$\delta_1(T) = \left\{\begin{array}{c} \frac{1}{8}T^2\ln\left(1+\frac{1}{4T^2}\right)+\frac{3}{32}\ln\left(1+4T^2\right)+\frac{1}{4}T\cdot\tan^{-1}\left(\frac{1}{2T}\right) \\ +\int_0^\infty \overset{\vee}{q}_0(u)\ln\left((u+\frac{1}{4})^2+\frac{T^2}{4}\right)\mathrm{d}u+2A \end{array}\right\} \qquad (76)$$

$$= \frac{1}{48}\ln T + C_1 + O(\frac{1}{T^2}) \qquad (77)$$

where

$$C_1 = \frac{5}{32}+\frac{17}{48}\ln 2+2A\quad. \qquad (78)$$

Combining equations 69 and 77 we then get the following equation which we will use in the $\mu = -1$ calculation:

$$\frac{1}{\pi}\left\{T\delta_0(T) - \delta_1(T)\right\} = \frac{1}{\pi}\left\{-\frac{1}{48}\ln(T) + (\frac{1}{48} - C_1) + O(\frac{1}{T^2})\right\} \quad . \tag{79}$$

Césaro calculation for $\delta_2(T)$: As for $\delta_2(T)$, from equation 76 we have

$$\delta_2(T) = \left\{ \begin{array}{c} \frac{1}{8}\int_0^T t^2 \ln\left(1 + \frac{1}{4t^2}\right) \mathrm{d}t + \frac{3}{32}\int_0^T \ln\left(1 + 4t^2\right) \mathrm{d}t \\ +\frac{1}{4}\int_0^T t \cdot \tan^{-1}\left(\frac{1}{2t}\right) \mathrm{d}t \\ +\int_0^T \int_0^\infty \overset{\vee}{q}_0(u) \ln\left((u + \frac{1}{4})^2 + \frac{t^2}{4}\right) \mathrm{d}u\mathrm{d}t + 2AT \end{array} \right\} \tag{80}$$

where the constant A is defined as per equation 75. Elementary integration again gives us that

$$\begin{aligned} \frac{1}{8}\int_0^T t^2 \ln\left(1 + \frac{1}{4t^2}\right) \mathrm{d}t &= \frac{1}{8} \cdot \frac{T^3}{3} \ln\left(1 + \frac{1}{4T^2}\right) + \frac{1}{12}\int_0^T \frac{t^2}{1 + 4t^2}\mathrm{d}t \\ &= \frac{1}{8} \cdot \frac{T^3}{3} \ln\left(1 + \frac{1}{4T^2}\right) + \frac{1}{48}T - \frac{1}{96}\tan^{-1}(2T) \end{aligned} \tag{81}$$

and

$$\begin{aligned} \frac{3}{32}\int_0^T \ln\left(1 + 4t^2\right) \mathrm{d}t &= \frac{3}{32}T \ln\left(1 + 4T^2\right) - \frac{3}{4}\int_0^T \frac{t^2}{1 + 4t^2}\mathrm{d}t \\ &= \frac{3}{32}T \ln\left(1 + 4T^2\right) - \frac{3}{16}T + \frac{3}{32}\tan^{-1}(2T) \end{aligned} \tag{82}$$

and

$$\begin{aligned} \frac{1}{4}\int_0^T t \cdot \tan^{-1}\left(\frac{1}{2t}\right) \mathrm{d}t &= \frac{1}{8}T^2 \tan^{-1}\left(\frac{1}{2T}\right) + \frac{1}{16}\int_0^T \frac{4t^2}{1 + 4t^2}\mathrm{d}t \\ &= \frac{1}{8}T^2 \tan^{-1}\left(\frac{1}{2T}\right) + \frac{1}{16}T - \frac{1}{32}\tan^{-1}(2T) \, . \end{aligned} \tag{83}$$

And, reversing the order of integration in the final integral, we get

$$\begin{aligned} &\int_0^\infty \overset{\vee}{q}_0(u) \int_0^T \ln\left((u + \frac{1}{4})^2 + \frac{t^2}{4}\right) \mathrm{d}t\mathrm{d}u \\ = &\int_0^\infty \overset{\vee}{q}_0(u) \left\{T \ln\left((u + \frac{1}{4})^2 + \frac{T^2}{4}\right) - 2\int_0^T \frac{t^2}{t^2 + 4(u + \frac{1}{4})^2}\mathrm{d}t\right\} \mathrm{d}u \end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty \overset{\vee}{q}_0(u) \left\{ \begin{array}{c} T\,\ln\left((u+\frac{1}{4})^2+\frac{T^2}{4}\right)-2T \\ \\ +8(u+\frac{1}{4})^2\int_0^T \frac{1}{t^2+4(u+\frac{1}{4})^2}\mathrm{d}t \end{array} \right\} \mathrm{d}u \\
&= \int_0^\infty \overset{\vee}{q}_0(u) \left\{ \begin{array}{c} T\,\ln\left((u+\frac{1}{4})^2+\frac{T^2}{4}\right)-2T \\ \\ +4(u+\frac{1}{4})\tan^{-1}\left(\frac{T}{2(u+\frac{1}{4})}\right) \end{array} \right\} \mathrm{d}u \\
&= \int_0^\infty \overset{\vee}{q}_0(u) \left\{ \begin{array}{c} T\left[2(\ln T-\ln 2)+\frac{4(u+\frac{1}{4})^2}{T^2}+\ldots\right]-2T \\ \\ +4(u+\frac{1}{4})\left[\frac{\pi}{2}-\frac{2(u+\frac{1}{4})}{T}+\ldots\right] \end{array} \right\} \mathrm{d}u \\
&= \left\{ \begin{array}{c} 2\left\{\int_0^\infty \overset{\vee}{q}_0(u)\mathrm{d}u\right\} T\ln T \\ \\ -(2+2\ln 2)\left\{\int_0^\infty \overset{\vee}{q}_0(u)\mathrm{d}u\right\} T \\ \\ +2\pi\left\{\int_0^\infty \overset{\vee}{q}_0(u)\cdot(u+\frac{1}{4})\mathrm{d}u\right\}+O(\frac{1}{T}) \end{array} \right\}
\end{aligned} \tag{84}
$$

Now we saw before that we have the Césaro integral

$$
\int_0^\infty \overset{\vee}{q}_0(u)\mathrm{d}u = \underset{k\to\infty}{Clim}\int_0^{k+\alpha}\overset{\vee}{q}_0(u)\mathrm{d}u = -\frac{1}{12} \quad . \tag{85}
$$

In the same way we have

$$
\begin{aligned}
& \int_0^{k+\alpha} \overset{\vee}{q}_0(u)\cdot(u+\frac{1}{4})\mathrm{d}u \\
= \;& \sum_{j=0}^{k-1}\int_0^1(\tilde{\alpha}-\frac{1}{2})(j+\tilde{\alpha}+\frac{1}{4})\mathrm{d}\tilde{\alpha}+\int_0^\alpha(\tilde{\alpha}-\frac{1}{2})(k+\tilde{\alpha}+\frac{1}{4})\mathrm{d}\tilde{\alpha} \\
= \;& \sum_{j=0}^{k-1}\int_0^1(\tilde{\alpha}^2-\frac{1}{2}\tilde{\alpha})\mathrm{d}\tilde{\alpha}+(k+\frac{1}{4})\int_0^\alpha(\tilde{\alpha}-\frac{1}{2})\mathrm{d}\tilde{\alpha}+\int_0^\alpha(\tilde{\alpha}^2-\frac{1}{2}\tilde{\alpha})\mathrm{d}\tilde{\alpha} \\
= \;& \frac{1}{12}k+(k+\frac{1}{4})(\frac{1}{2}\alpha^2-\frac{1}{2}\alpha)-\frac{1}{4}\alpha^2+\frac{1}{3}\alpha^3 \\
= \;& \frac{1}{2}k\alpha^2-\frac{1}{2}k\alpha+\frac{1}{12}k+(\frac{1}{3}\alpha^3-\frac{1}{8}\alpha^2-\frac{1}{8}\alpha)
\end{aligned} \tag{86}
$$

and so, by equation 70, we also have, after simplification, the Césaro integral

$$
\int_0^\infty \overset{\vee}{q}_0(u)\cdot(u+\frac{1}{4})\mathrm{d}u = \underset{k\to\infty}{Clim}\int_0^{k+\alpha}\overset{\vee}{q}_0(u)\cdot(u+\frac{1}{4})\mathrm{d}u = -\frac{1}{48} \quad . \tag{87}
$$

Combining equations 85 and 87 in equation 84 it then follows that, up to

$O(\frac{1}{T})$, we have

$$\int_0^T \int_0^\infty \overset{\vee}{q}_0(u) \ln \left((u+\frac{1}{4})^2 + \frac{t^2}{4} \right) \mathrm{d}u\mathrm{d}t = -\frac{1}{6} T \ln T + \frac{1}{6}(1+\ln 2)T - \frac{\pi}{24} \quad . \tag{88}$$

Finally, combining equations 81, 82, 83 and 88 in equation 80 we get, after simplification, that

$$\delta_2(T) \quad = \quad \frac{1}{48} T \ln T + BT - \frac{\pi}{64} + O(\frac{1}{T}) \tag{89}$$

where

$$B = \frac{13}{96} + \frac{17}{48} \ln 2 + 2A \quad . \tag{90}$$

Combining equation 89 with our earlier expressions for $\delta_0(T)$ and $\delta_1(T)$ in equations 69 and 77, we then get the following equation which we will use in the $\mu = -2$ calculation:

$$\begin{aligned} \frac{1}{\pi} \left\{ \begin{array}{c} T^2\delta_0(T) - 2T\delta_1(T) \\ \\ +2\delta_2(T) \end{array} \right\} &= \frac{1}{\pi} \left\{ \begin{array}{c} \frac{1}{48}T - \frac{1}{24} T \ln T - 2C_1 T \\ \\ +\frac{1}{24} T \ln T + 2BT - \frac{\pi}{32} \end{array} \right\} + O(\frac{1}{T}) \\ &= \frac{1}{\pi} \left\{ DT - \frac{\pi}{32} \right\} + O(\frac{1}{T}) \end{aligned} \tag{91}$$

where

$$D = \frac{1}{48} - 2C_1 + 2B = -\frac{1}{48} \, . \tag{92}$$

4.2.4 Extending results for the $S(t)$-pieces of $r_{NT}(s_0, \mu)$

We now turn to $S_1(T)$ and $S_2(T)$, where we are no longer in the realm of closed-form formulae, but rather just seeking Césaro limits as $T \to \infty$. We saw from Result 1 that $S(T) = O(logT)$ and $S_1(T) = O(logT)$ and hence that $S(T)$ is strongly Césaro-convergent to zero via one application of P. But things are much harder for the higher $S_i(T)$ because there are no unconditional estimates on $\int_0^T S_i(t)\mathrm{d}t$ analogous to equation 30.

However, by [10] (see section 14.13, equations 14.13.1, 14.13.2 and 14.13.8) we find that if we assume RH (which we are doing here), then such estimates do exist providing we are careful. Specifically, we have the following:

Result 3: *Assuming RH, we obtain improved estimates on S and S_1, namely that*

$$S(T) = O\left(\frac{logT}{loglogT}\right) \quad \text{and} \quad S_1(T) = O\left(\frac{logT}{(loglogT)^2}\right) \qquad \text{as} \quad T \to \infty \, . \tag{93}$$

Moreover, writing $S_0^*(T) := S(T)$ there exists a unique chain of anti-derivatives $\{S_n^*(T)\}_{n=0}^\infty$ such that $\frac{\mathrm{d}}{\mathrm{d}t} S_{n+1}^*(T) = S_n^*(T)$ and

$$S_n^*(T) = O\left(\frac{logT}{(loglogT)^{n+1}}\right) \qquad \text{as} \quad T \to \infty \tag{94}$$

for all $n \in \mathbb{Z}_{\geq 0}$.

Comments: **(i)** Note in this result that for $n \geq 2$ we do *not* have that $S_n^*(T) = S_n(T)$ as functions. This is because the $S_n(T)$ are explicitly defined as a chain of anti-derivatives obtained by integrating the previous one from lower limit 0 to T, whereas the $S_n^*(T)$ are defined by being the unique chain of anti-derivatives satisfying the estimates in equation 94 for all n.

Since S_1 and S_1^* are both anti-derivatives of S they differ only by a constant and so both would still satisfy equation 94. But $S_2(T)$ would then fail to satisfy equation 94 since this constant difference at the $n = 1$ level would integrate to a linear divergence at $n = 2$ - and this difference would then grow to a quadratic difference, then cubic etc as we moved through $n = 3, 4, \ldots$. Effectively, one way we can think of the $S_n^*(T)$ is as a chain of integrals like the $S_n(T)$, but where the lower integration-limit each time has been precisely selected so as to ensure that we can retain the truth of estimate 94 not just for that level of n, but also for all subsequent levels in the chain.[7]

(ii) The relationship between the $S_n(T)$ and the $S_n^*(T)$ is easy to make explicit and will be useful. We have $S_0 = S = S_0^*$, but then each integration introduces a potential difference consisting of a constant of integration, $C_S^{(n)}$ ($n \in \mathbb{Z}_{\geq 1}$), so that we have

$$S_1(T) = S_1^*(T) + C_S^{(1)} \tag{95}$$

$$S_2(T) = S_2^*(T) + C_S^{(1)} T + C_S^{(2)} \tag{96}$$

$$S_3(T) = S_3^*(T) + \frac{1}{2!} C_S^{(1)} T^2 + C_S^{(2)} T + C_S^{(3)}$$

and in general, if we set $C_S^{(0)} := 0$, then

$$S_n(T) = S_n^*(T) + \sum_{i=0}^{n} \frac{1}{(n-i)!} C_S^{(i)} T^{n-i}. \tag{97}$$

(iii) The chain $\{S_n^*(t)\}_{n=0}^{\infty}$ forms a familiar quantity within the generalised Césaro framework, namely a Césaro-adapted scale (see [III] for definition and discussion). This makes it very useful for generalised Césaro computation because, in the same way we established the corresponding result in [III] for the periodic Césaro-adapted scale $\left\{\overset{\vee}{q}_n(X)\right\}_{n=0}^{\infty}$, the following result is readily deduced by induction using integration by parts:

[7] In [10] this distinction is elided and equation 14.13.8 is quoted as holding for anti-derivatives $S_n(T)$. This certainly caused the author of this paper confusion for a long time, as I erroneously assumed that the subsequent $S_n(T)$ ($n \geq 2$) were, like $S_1(T)$, to be understood as integrals of the preceding $S_{n-1}(T)$ from lower limit 0 to T. In hindsight this assumption was, of course, silly; but it was an easy error to make - and one with very significant ramifications - so we find it worthwhile to make the distinction clearer here.

Result 4: *For all $r, n \in \mathbb{Z}_{\geq 0}$ we have the strong Césaro asymptotic relationship that*

$$T^n S_r^*(T) \overset{C}{\simeq} 0 \qquad \text{as} \quad T \to \infty \tag{98}$$

via P^{n+1}, i.e.

$$P^{n+1}\left[t^n S_r^*(t)\right](T) = o(1)\,. \tag{99}$$

Combining this result with equation 97 we can readily calculate the Césaro limit of each $S_n(T)$, albeit that it is only for $n = 0$ and $n = 1$ that this is a strong Césaro limit; for $n \in \mathbb{Z}_{\geq 2}$ it will require removing powers of z (resp. $\tilde{z}$) as generalised eigenfunctions of P, owing to the powers of T in equation 97.

The two key results which we will need for the $\mu = -1$ and $\mu = -2$ root identity calculations, however, *are* both strong Césaro asymptotic results. By result 4, for the $\mu = -1$ case we have that

$$TS(T) - S_1(T) \;=\; TS_0^*(T) - S_1^*(T) - C_S^{(1)} \;\to\; -C_S^{(1)} \tag{100}$$

via application of P^2; while for $\mu = -2$ we have that

$$\begin{aligned} T^2 S(T) - 2TS_1(T) + 2S_2(T) \quad &= \quad T^2 S_0^*(T) - 2TS_1^*(T) + 2S_2^*(T) + 2C_S^{(2)} \\ &\to \quad 2C_S^{(2)} \end{aligned} \tag{101}$$

via application of P^3.

4.2.5 Calculation of $r_\zeta(s_0, -1)$ and test of the root identity at $\mu = -1$

We have now completed our preliminaries and are finally in a position to perform our test of the $\mu = -1$ root identity.

In equation 13 we have $r_P(s_0, -1) = s_0 - 1$, and by equation 61, $r_T(s_0, -1) = \frac{1}{4}s_0^2 + \frac{1}{2}s_0 + \frac{1}{6}$.

And when $\mu = -1$, in equations 22 and 40 we get that the partial sum for $r_{NT_+}(s_0, -1) + r_{NT_-}(s_0, -1)$ is given by

$$s_{+,-}(s_0, -1; T, \tilde{T}) \quad = \quad \left\{ \begin{array}{c} i\left\{\sum_{\{\gamma_i < T\}} M_i \gamma_i + 2\pi q i \sum_{\{\gamma_i < T\}} M_i \right\} \\ \\ -i\left\{\sum_{\{\tilde{\gamma}_i < \tilde{T}\}} M_i \tilde{\gamma}_i - 2\pi q i \sum_{\{\tilde{\gamma}_i < \tilde{T}\}} M_i \right\} \end{array} \right\} + o(1) \tag{102}$$

Now from section 4.1, formula 53, we know that

$$\sum_{\{\gamma_i < T\}} M_i + \sum_{\{\tilde{\gamma}_i < \tilde{T}\}} M_i \quad \overset{C}{\sim} \quad \left\{ \begin{array}{c} iw \ln w - (\frac{\pi}{2} + i)w \\ \\ -i\tilde{w} \ln \tilde{w} - (\frac{\pi}{2} - i)\tilde{w} + iq \ln\left(\frac{\tilde{w}}{w}\right) \end{array} \right\} + (\frac{7}{4} + \pi q) \tag{103}$$

On the other hand,

$$\sum_{\{\gamma_i<T\}} M_i\gamma_i - \sum_{\{\tilde\gamma_i<\tilde T\}} M_i\tilde\gamma_i = \int_0^T t\mathrm{d}N(t) - \int_0^{\tilde T} \tilde t\mathrm{d}N(\tilde t)$$

$$= \left\{ \begin{array}{c} \left\{\int_0^T t\mathrm{d}\check N(t) + \int_0^T t\mathrm{d}S(t) + \frac{1}{\pi}\int_0^T t\mathrm{d}\delta(t)\right\} \\ -\left\{\int_0^{\tilde T} \tilde t\mathrm{d}\check N(\tilde t) + \int_0^{\tilde T} \tilde t\mathrm{d}S(\tilde t) + \frac{1}{\pi}\int_0^{\tilde T} \tilde t\mathrm{d}\delta(\tilde t)\right\} \end{array} \right\}$$

$$= \left\{ \begin{array}{c} \left\{T\check N(T) - \check N_1(T)\right\} - \left\{\tilde T\check N(\tilde T) - \check N_1(\tilde T)\right\} \\ +\frac{1}{\pi}\left\{T\delta(T) - \delta_1(T)\right\} - \frac{1}{\pi}\left\{\tilde T\delta(\tilde T) - \delta_1(\tilde T)\right\} \\ +\left\{TS(T) - S_1(T)\right\} - \left\{\tilde TS(\tilde T) - S_1(\tilde T)\right\} \end{array} \right\} \quad . \tag{104}$$

Now, by equation 100, both $TS(T)-S_1(T)$ and $\tilde TS(\tilde T)-S_1(\tilde T)$ strongly Césaro converge to $-C_S^{(1)}$ and these limiting contributions cancel. As for the terms involving $\check N$, by equation 66, we have that

$$T\check N(T) - \check N_1(T) = 2\pi\left\{\frac{1}{2}u^2\ln u - \frac{1}{4}u^2\right\}$$

and combining with the corresponding result for $\tilde T\check N(\tilde T) - \check N_1(\tilde T)$ we thus get

$$\left\{T\check N(T) - \check N_1(T)\right\} - \left\{\tilde T\check N(\tilde T) - \check N_1(\tilde T)\right\} = 2\pi\left\{ \begin{array}{c} \frac{1}{2}u^2\ln u - \frac{1}{4}u^2 \\ -\frac{1}{2}\tilde u^2\ln\tilde u + \frac{1}{4}\tilde u^2 \end{array} \right\} \tag{105}$$

Lastly, from result 79, we have

$$\frac{1}{\pi}\left\{T\delta(T) - \delta_1(T)\right\} = -\frac{1}{48\pi}\ln T + \frac{1}{48\pi} - \frac{1}{\pi}C_1 + o(1) \tag{106}$$

where C_1 is as given by equation 78. Combining with the corresponding result for $\frac{1}{\pi}\left\{\tilde T\delta(\tilde T) - \delta_1(\tilde T)\right\}$ and noting that $\ln\left(\frac{\tilde T}{T}\right) = \ln\left(\frac{\tilde u}{u}\right)$ we get that

$$\frac{1}{\pi}\left\{T\delta(T) - \delta_1(T)\right\} - \frac{1}{\pi}\left\{\tilde T\delta(\tilde T) - \delta_1(\tilde T)\right\} = \frac{1}{48\pi}\ln\left(\frac{\tilde u}{u}\right) + o(1) \quad . \tag{107}$$

We now need to change variables from $u, \tilde u$ to the geometric variables $w, \tilde w$.

Noting that $\ln\left(\frac{\tilde{u}}{u}\right) = -i\pi + \ln\left(\frac{\tilde{w}}{w}\right) + o(1)$ and simplifying, we obtain that

$$\sum_{\{\gamma_i<T\}} M_i\gamma_i - \sum_{\{\tilde{\gamma}_i<\tilde{T}\}} M_i\tilde{\gamma}_i \;=\; 2\pi \left\{ \begin{array}{c} -\frac{1}{2}w^2\ln w - (\frac{i\pi}{4}-\frac{1}{4})w^2 + qw\ln w \\ +\frac{i\pi}{2}qw - \frac{1}{2}q^2\ln w - \frac{i\pi}{4}q^2 \\ +\frac{1}{2}\tilde{w}^2\ln\tilde{w} - (\frac{i\pi}{4}+\frac{1}{4})\tilde{w}^2 - q\tilde{w}\ln\tilde{w} \\ +\frac{i\pi}{2}q\tilde{w} + \frac{1}{2}q^2\ln\tilde{w} - \frac{i\pi}{4}q^2 \\ -\frac{i}{96\pi} + \frac{1}{96\pi^2}\ln\left(\frac{\tilde{w}}{w}\right) \end{array} \right\} \tag{108}$$

and, after combining equations 103 and 108 in equation 102 and further simplifying, that

$$s_{+,-}(s_0,-1;T,\tilde{T}) \;\overset{C}{\sim}\; \left\{ \begin{array}{c} -2\pi i \left\{ \begin{array}{c} \frac{1}{2}w^2\ln w + (\frac{i\pi}{4}-\frac{1}{4})w^2 - qw \\ -\frac{1}{2}\tilde{w}^2\ln\tilde{w} + (\frac{i\pi}{4}+\frac{1}{4})\tilde{w}^2 + q\tilde{w} \\ +\left(\frac{1}{2}q^2 - \frac{1}{96\pi^2}\right)\ln\left(\frac{\tilde{w}}{w}\right) \end{array} \right\} \\ -2\pi i\left(-iq(\frac{7}{4}+\pi q) + \frac{i\pi}{2}q^2 + \frac{i}{96\pi}\right) \end{array} \right\}. \tag{109}$$

Noting that the non-trivial eigenfunctions and generalised eigenfunctions of P have zero Césaro limit, and invoking result 2 again[8], we finally obtain that

$$\begin{aligned} s_{+,-}(s_0,-1;T,\tilde{T}) \;&\overset{C}{\to}\; 2\pi i\left\{i\cdot\frac{7}{4}q + \frac{i\pi}{2}q^2 - \frac{i}{96\pi}\right\} \\ &= -\frac{1}{4}s_0^2 - \frac{3}{2}s_0 + \frac{5}{6} \end{aligned} \tag{110}$$

and thus

$$r_{NT_+}(s_0,-1) + r_{NT_-}(s_0,-1) = -\frac{1}{4}s_0^2 - \frac{3}{2}s_0 + \frac{5}{6}\,. \tag{111}$$

But then, combining our results for $r_{NT_+}(s_0,-1)+r_{NT_-}(s_0,-1)$, $r_T(s_0,-1)$ and $r_P(s_0,-1)$ in equation 8 we deduce at last that we do indeed have

Lemma 3:

$$r_\zeta(s_0,-1) = 0 \qquad \text{for arbitrary } Re(s_0) > 1 \tag{112}$$

and so ζ does also satisfy the generalised root identities at $\mu = -1$.

[8] note that the final divergent expression in set brackets in equation 109 is again equal to the limiting divergent piece as $\mu \to -1$ arising from equations 37 and 42

Comment: When $\mu = -1$, the formula for $r_{NT}(s_0, \mu)$ involves taking a *difference* of contributions from the NT_+ and NT_- halves of the non-trivial roots. As in the case of $\mu = 0$, this plays an important role in allowing geometry to successfully adjust the contributions from w-terms and $\tilde{w}$-terms in our Césaro computations arising from the $\check{N}(T)$-pieces and the $\delta(T)$-pieces.

However, it leads to *cancellation* of the limiting $C_S^{(1)}$ contributions from the $S(T)$-pieces, so that we get no further constraints on the behaviour of $S(T)$ from the $\mu = -1$ root identity. In the $\mu = -2$ root identity considered next, however, we take *sums*, not differences, of NT_+ and NT_- contributions and so cancellation no longer occurs. This leads to a first new result regarding $S(T)$.

4.3 The case of $\mu = -2$

In this case in equation 13 we have $r_P(s_0, -2) = -(s_0 - 1)^2$, while by equation 62 we have $r_T(s_0, -2) = -\frac{1}{6}s_0^3 - \frac{1}{2}s_0^2 - \frac{1}{3}s_0$.

And when $\mu = -2$ we get from equations 22 and 40 that the partial sum for $r_{NT_+}(s_0, -2) + r_{NT_-}(s_0, -2)$ is given by

$$s_{+,-}(s_0, -2; T, \tilde{T}) = \left\{ \begin{array}{c} -\left\{ \begin{array}{c} \sum_{\{\gamma_i < T\}} M_i\gamma_i^2 + 4\pi qi \sum_{\{\gamma_i < T\}} M_i\gamma_i \\ -4\pi^2q^2 \sum_{\{\gamma_i < T\}} M_i \end{array} \right\} \\ -\left\{ \begin{array}{c} \sum_{\{\tilde{\gamma}_i < \tilde{T}\}} M_i\tilde{\gamma}_i^2 - 4\pi qi \sum_{\{\tilde{\gamma}_i < \tilde{T}\}} M_i\tilde{\gamma}_i \\ -4\pi^2q^2 \sum_{\{\tilde{\gamma}_i < \tilde{T}\}} M_i \end{array} \right\} \end{array} \right\} + o(1) \tag{113}$$

Now from Section 4.1 we have formula 53 for $\sum_{\{\gamma_i < T\}} M_i + \sum_{\{\tilde{\gamma}_i < \tilde{T}\}} M_i$; and from subsection 4.2.5 we have formula 108 for $\sum_{\{\gamma_i < T\}} M_i\gamma_i - \sum_{\{\tilde{\gamma}_i < \tilde{T}\}} M_i\tilde{\gamma}_i$. As for the remaining terms, as in previous calculations we have

$$\begin{aligned} \sum_{\{\gamma_i < T\}} M_i\gamma_i^2 + \sum_{\{\tilde{\gamma}_i < \tilde{T}\}} M_i\tilde{\gamma}_i^2 &= \int_0^T t^2 \mathrm{d}N(t) + \int_0^{\tilde{T}} \tilde{t}^2 \mathrm{d}N(\tilde{t}) \\ &= \left\{ \begin{array}{c} \left\{\int_0^T t^2 \mathrm{d}\check{N}(t) + \int_0^T t^2 \mathrm{d}S(t) + \frac{1}{\pi}\int_0^T t^2 \mathrm{d}\delta(t)\right\} \\ + \left\{\int_0^{\tilde{T}} \tilde{t}^2 \mathrm{d}\check{N}(\tilde{t}) + \int_0^{\tilde{T}} \tilde{t}^2 \mathrm{d}S(\tilde{t}) + \frac{1}{\pi}\int_0^{\tilde{T}} \tilde{t}^2 \mathrm{d}\delta(\tilde{t})\right\} \end{array} \right\} \end{aligned}$$

$$= \left\{ \begin{array}{c} \left\{T^2\check{N}(T) - 2T\check{N}_1(T) + 2\check{N}_2(T)\right\} + \left\{\tilde{T}^2\check{N}(\tilde{T}) - 2\tilde{T}\check{N}_1(\tilde{T}) + 2\check{N}_2(\tilde{T})\right\} \\ +\frac{1}{\pi}\left\{T^2\delta(T) - 2T\delta_1(T) + 2\delta_2(T)\right\} + \frac{1}{\pi}\left\{\tilde{T}^2\delta(\tilde{T}) - 2\tilde{T}\delta_1(\tilde{T}) + 2\delta_2(\tilde{T})\right\} \\ + \left\{T^2S(T) - 2TS_1(T) + 2S_2(T)\right\} + \left\{\tilde{T}^2S(\tilde{T}) - 2\tilde{T}S_1(\tilde{T}) + 2S_2(\tilde{T})\right\} \end{array} \right\} \tag{114}$$

Now by equation 67 we have that

$$T^2\check{N}(T) - 2T\check{N}_1(T) + 2\check{N}_2(T) = (2\pi)^2\left\{\frac{1}{3}u^3\ln u - \frac{1}{9}u^3\right\}$$

and combining with the corresponding result for $\tilde{T}^2\check{N}(\tilde{T}) - 2\tilde{T}\check{N}_1(\tilde{T}) + 2\check{N}_2(\tilde{T})$ we thus get that

$$\left\{\begin{array}{c}\left\{T^2\check{N}(T) - 2T\check{N}_1(T) + 2\check{N}_2(T)\right\} + \\ \\ \left\{\tilde{T}^2\check{N}(\tilde{T}) - 2\tilde{T}\check{N}_1(\tilde{T}) + 2\check{N}_2(\tilde{T})\right\}\end{array}\right\} = (2\pi)^2\left\{\begin{array}{c}\frac{1}{3}u^3\ln u - \frac{1}{9}u^3 \\ \\ +\frac{1}{3}\tilde{u}^3\ln\tilde{u} - \frac{1}{9}\tilde{u}^3\end{array}\right\}. \tag{115}$$

And from result 91, we have that

$$\frac{1}{\pi}\left\{T^2\delta(T) - 2T\delta_1(T) + 2\delta_2(T)\right\} = -\frac{1}{48\pi}T - \frac{1}{32} + o(1) \tag{116}$$

so that, along with the corresponding result for $\frac{1}{\pi}\left\{\tilde{T}^2\delta(\tilde{T}) - 2\tilde{T}\delta_1(\tilde{T}) + 2\delta_2(\tilde{T})\right\}$ we obtain that

$$\frac{1}{\pi}\left\{\begin{array}{c}T^2\delta(T) - 2T\delta_1(T) + 2\delta_2(T) + \\ \\ \tilde{T}^2\delta(\tilde{T}) - 2\tilde{T}\delta_1(\tilde{T}) + 2\delta_2(\tilde{T})\end{array}\right\} = -\frac{1}{24}u - \frac{1}{24}\tilde{u} - \frac{1}{16} + o(1) \quad . \tag{117}$$

Lastly, by equation 101, both $T^2S(T) - 2TS_1(T) + 2S_2(T)$ and $\tilde{T}^2S(\tilde{T}) - 2\tilde{T}S_1(\tilde{T}) + 2S_2(\tilde{T})$ strongly Césaro converge to $2C_S^{(2)}$. But these limiting contributions now reinforce rather than cancel, so that

$$\left\{\begin{array}{c}\left\{T^2S(T) - 2TS_1(T) + 2S_2(T)\right\} + \\ \\ \left\{\tilde{T}^2S(\tilde{T}) - 2\tilde{T}S_1(\tilde{T}) + 2S_2(\tilde{T})\right\}\end{array}\right\} \overset{C}{\simeq} 4C_S^{(2)}\,. \tag{118}$$

It follows overall that, in terms of w and $\tilde{w}$, we have

$$\sum_{\{\gamma_i<T\}} M_i\gamma_i^2 + \sum_{\{\tilde{\gamma}_i<\tilde{T}\}} M_i\tilde{\gamma}_i^2$$

$$\overset{C}{\simeq} \quad (2\pi)^2\left\{\begin{array}{c}-\frac{i}{3}w^3\ln w + (\frac{\pi}{6}+\frac{i}{9})w^3 + iqw^2\ln w - \frac{\pi}{2}qw^2 \\ \\ -iq^2w\ln w + (\frac{\pi q^2}{2} - \frac{iq^2}{2} - \frac{i}{96\pi^2})w - \frac{\pi}{6}q^3 \\ \\ +\frac{i}{3}\tilde{w}^3\ln\tilde{w} + (\frac{\pi}{6}-\frac{i}{9})\tilde{w}^3 - iq\tilde{w}^2\ln\tilde{w} - \frac{\pi}{2}q\tilde{w}^2 \\ \\ +iq^2\tilde{w}\ln\tilde{w} + (\frac{\pi q^2}{2} + \frac{iq^2}{2} + \frac{i}{96\pi^2})\tilde{w} - \frac{\pi}{6}q^3 \\ \\ -\frac{i}{3}q^3\ln\left(\frac{\tilde{w}}{w}\right) - \frac{1}{64\pi^2}\end{array}\right\} + 4C_S^{(2)} \tag{119}$$

Combining the results for $\sum_{\{\gamma_i<T\}} M_i + \sum_{\{\tilde{\gamma}_i<\tilde{T}\}} M_i$ and $\sum_{\{\gamma_i<T\}} M_i\gamma_i - \sum_{\{\tilde{\gamma}_i<\tilde{T}\}} M_i\tilde{\gamma}_i$ with this result in equation 113 we then finally get, after extensive simplification, that

$$
\begin{aligned}
& s_{+,-}(s_0,-2;T,\tilde{T}) \\
\overset{C}{\sim} \quad & (2\pi)^2 \left\{ \begin{array}{c} i \left\{ \begin{array}{c} \frac{1}{3}w^3\ln w + (\frac{i\pi}{6}-\frac{1}{9})w^3 - \frac{q}{2}w^2 \\ + \left(-\frac{1}{2}q^2 + \frac{1}{96\pi^2}\right) w \\ -\frac{1}{3}\tilde{w}^3\ln\tilde{w} + (\frac{i\pi}{6}+\frac{1}{9})\tilde{w}^3 + \frac{q}{2}\tilde{w}^2 \\ + \left(\frac{1}{2}q^2 - \frac{1}{96\pi^2}\right)\tilde{w} \\ + \left(\frac{1}{3}q^3 - \frac{1}{48\pi^2}q\right)\ln\left(\frac{\tilde{w}}{w}\right) \end{array} \right\} \\ + \left((\frac{7}{4}+\pi q)q^2 - \frac{2\pi}{3}q^3 - \frac{1}{48\pi}q + \frac{1}{64\pi^2}\right) \end{array} \right\} - 4C_S^{(2)}
\end{aligned}
\tag{120}
$$

The non-trivial eigenfunctions and generalised eigenfunctions of P have zero Césaro limit as usual and so, invoking result 2 once more as before, we deduce that

$$
\begin{aligned}
s_{+,-}(s_0,-2;T,\tilde{T}) \quad & \overset{C}{\to} \quad (2\pi)^2 \left\{ \begin{array}{c} \frac{(\frac{1}{2}s_0+\frac{3}{2})(s_0-\frac{1}{2})^2}{4\pi^2} - \frac{2}{3}\frac{(s_0-\frac{1}{2})^3}{8\pi^2} \\ -\frac{(s_0-\frac{1}{2})}{96\pi^2} + \frac{1}{64\pi^2} \end{array} \right\} - 4C_S^{(2)} \\
& = \quad \frac{1}{6}s_0^3 + \frac{3}{2}s_0^2 - \frac{5}{3}s_0 + \frac{1}{2} - 4C_S^{(2)}
\end{aligned}
\tag{121}
$$

and thus $r_{NT_+}(s_0,-2) + r_{NT_-}(s_0,-2) = \frac{1}{6}s_0^3 + \frac{3}{2}s_0^2 - \frac{5}{3}s_0 + \frac{1}{2} - 4C_S^{(2)}$.

But then finally, combining our results for $r_{NT_+}(s_0,-2) + r_{NT_-}(s_0,-2)$, $r_P(s_0,-2)$ and $r_T(s_0,-2)$ in equation 8 we find that, unlike in the previous cases, we do not automatically have $r_\zeta(s_0,-2) = 0$. Rather, while all terms of degree greater than or equal to 1 in s_0 cancel, we are left with a residual constant term, namely

$$
r_\zeta(s_0,-2) = -\frac{1}{2} - 4C_S^{(2)} \qquad \text{for arbitrary } Re(s_0) > 1 \quad . \tag{122}
$$

In order for ζ to satisfy the generalised root identity at $\mu = -2$ this must be zero, by equation 7, and we thus immediately obtain the following result (conditional on RH):

Result 5: *ζ satisfies the generalised root identity at $\mu = -2$ if and only if $C_S^{(2)} = -\frac{1}{8}$; or equivalently, if and only if we have the Césaro integral identity*

$$
\int_0^\infty t^2\,\mathrm{d}S(t) = -\frac{1}{4}\,. \tag{123}
$$

Comments: **(i)** The latter form of the equivalence follows immediately from equation 101 and is in fact a strong Césaro asymptotic relationship, meaning that $\int_0^T t^2 \, \mathrm{d}S(t) \overset{C}{\simeq} -\frac{1}{4}$ as $T \to \infty$ via P^3. It can be viewed either as a new condition on the oscillatory behaviour of $S(t)$, or else as a new integral identity evaluating the Mellin transform of $S'(t)$ at the specific transform-value $s = 3$.

In a future series of papers we will explore in depth how the generalised Césaro framework is in fact the right framework for the understanding and analysis of Mellin transforms in general. This identity - and corresponding ones for $\int_0^\infty t^{2n} \, \mathrm{d}S(t)$, $n \in \mathbb{Z}_{\geq 2}$ which we will derive in the third paper in this set on ζ - will then be found to give insight (conditional on RH) on the asymptotic behaviour of $S(T)$ for T near ∞.

(ii) In this paper we have presented strong evidence that the zeta function does indeed satisfy the generalised root identities at all $\mu \in \mathbb{C} \setminus \{1\}$ for $Re(s_0) > 1$. The evidence presented does not, however, constitute a rigorous proof and thus result 5 is, for now, conditional on ζ satisfying the generalised root identities at $\mu = -2$.

In the next paper in this set, we provide this rigorous proof, at which stage result 5 will become a fully-established theorem regarding $S(T)$, conditional on RH.

In the third and final paper in this series on zeta and the generalised root identities, we then return to exploration, extending result 5 to a *family* of new results imposing conditions on $S(T)$, conditional on RH. These are interpretable either as a countable set of new Césaro integral identities for $S(t)$, or as a general theorem regarding special-values of the Mellin-transform of $S'(t)$ with implications for the asymptotic behaviour of $S(t)$ as $t \to \infty$. We believe these results, of which Result 5 above is the first, are entirely new.

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