# Root Identities II: Root identities for $\zeta$ - Part B

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**Abstract**

In this paper we prove that $\zeta(s)$ satisfies the generalised root identities for $Re(s_0) > 1$ and for all $\mu \in \mathbb{C} \setminus \{1\}$. We also show (assuming the Riemann hypothesis) that for such $\mu$ and $s_0$, the analytic continuation of the root side of these identities to the region $Re(\mu) \leq 1$, where $r_\zeta(s_0, \mu)$ is classically divergent, is achieved by the generalised geometric Césaro framework. Finally we show that these Césaro methods give the correct values of the root side at the key points $\mu \in \mathbb{Z}_{\leq 0}$, with no "anomalies" or removable singularities at these points.

## 1 Introduction

In [VIII] we began an investigation of the generalised root identities (as introduced in [VII]) for the Riemann zeta function $\zeta$.

We used the Euler product formula to derive a general formula for the derivative side, $d_\zeta(s_0, \mu)$, which extends to an analytic function of $\mu$ and $s_0$ on this domain. We then turned to the root side and we first showed numerically that $\zeta$ appears to satisfy the root identities for arbitrary $Re(\mu) > 1$, where the series defining the root side are all classically convergent.

Assuming the Riemann hypothesis (RH), we then used generalised Césaro methods to analytically continue $r_\zeta(s_0, \mu)$ to $Re(\mu) \leq 1$ where these series become divergent. We thereby extended our numerical tests to show continued equality of $d_\zeta(s_0, \mu)$ and $r_\zeta(s_0, \mu)$ for such $\mu$ (and $Re(s_0) > 1$ initially).

In the course of this analysis, we showed that contributions to $r_\zeta(s_0, \mu)$ from the non-trivial roots with positive imaginary parts, and from their conjugates with negative imaginary parts, separately lead to simple poles in $\mu$ at the non-positive integer points in the $\mu$-plane; but that these poles cancel off, leaving $r_\zeta(s_0, \mu)$ analytic and well-defined across these points. We demonstrated this numerically and also explained it analytically. In doing so, we showed that terms of the form $\ln\left(\frac{\bar{w}}{w}\right)$, which occur in the p-sums for $r_\zeta(s_0, \mu)$ *at* such non-positive integer $\mu$-values, should be ascribed zero Césaro limit within the generalised Césaro framework even though pure log-divergences are usually problematic.

Finally, with this in mind, we considered the first few such non-positive integer $\mu$-values, where exact, not merely numerical, computation is possible on the root side. We confirmed that $\zeta$ continues to satisfy the generalised root identities at $\mu = 0$ and $\mu = -1$. However, at $\mu = -2$, we showed that $\zeta$ continues to satisfy the generalised root identity if and only if a first new condition on the argument of the zeta function, $S(t)$, is satisfied. Specifically, $S(t)$ must satisfy $\underset{T\to\infty}{Clim} \int_0^T t^2 \, \mathrm{d}S(t) = -\frac{1}{4}$ where this is a strong Césaro limit via application of $P^3$ (i.e. three averagings in $T$). Further such new conditions on $S(T)$ would be implied by $\zeta$ satisfying the generalised root identities at $\mu = -3, -4, \ldots$.

The combination of numerical and exact checks in [VIII] provided convincing evidence that $\zeta$ does indeed satisfy the generalised root identities at all $\mu \in \mathbb{C} \setminus \{1\}$. It did not, however, provide a rigorous *proof* of this proposition. The purpose of this paper is to provide this missing proof; and secondly, to provide a brief rigourisation of the arguments given in [VIII] justifying the correctness of generalised Césaro methods in analytically continuing $r_\zeta(s_0, \mu)$ from its region of convergence to $Re(\mu) \leq 1$.

Some readers - those who are not overly fastidious regarding meticulous proof, or who would prefer to move quickly to new territories and results, rather than take a detour into the tilled fields of "theorem-lemma-proof" - may be satisfied with the evidence given in [VIII]. If so, they should feel free to take these propositions as established and, with the author's good wishes, skip heedlessly on to the next paper where, on the basis of them, we derive the whole family of new results hinted at above regarding $S(T)$.

For those opting to remain for this leg of the carriage-trip, however, we provide in section 2 the proof that $\zeta$ satisfies the generalised root identities for $\mu \in \mathbb{C} \setminus \{1\}$ and $Re(s_0) > 1$; and we provide in section 3 the review of generalised Césaro methods on the root-side, and the justification of their applicability.

In more detail, in subsection 2.1 we recall the general formula for $d_\zeta(s_0, \mu)$ which we derived in [VIII] and which extends directly to an analytic function on all $\mu \in \mathbb{C} \setminus \{1\}$ for $Re(s_0) > 1$. To prove that $\zeta$ satisfies the generalised root identities in general it therefore suffices to prove that $d_\zeta(s_0, \mu) = r_\zeta(s_0, \mu)$ on, for example, a $\mu$-interval where they are both well-defined, since then they must co-oincide in this region and continue to do so outside it by the uniqueness of analytic continuation. In section 2.2 we provide this proof on the $\mu$-interval $(\frac{3}{2}, 2)$. We use fractional calculus to analyse the expression for $d_\zeta(s_0, \mu)$ and show its equivalence to $r_\zeta(s_0, \mu)$ there.

In section 3 we then briefly rigourise the analysis supplied in [VIII] showing that for $Re(s_0) > 1$, the analytic continuation of $r_\zeta(s_0, \mu)$ to $Re(\mu) \leq 1$ ($\mu \neq 1$) is accomplished by generalised geometric Césaro analysis. This Césaro analysis covers the contribution of the trivial roots ($r_T(s_0, \mu)$), and each of the three components (arising from $\check{N}$, $\delta$ and $S$) of the contribution of the non-trivial roots ($r_{NT}(s_0, \mu)$) - and it has already been outlined in detail in [VIII]. We therefore only recapitulate that working very briefly here and add the necessary perspective to explain rigorously how the correctness of extension by Césaro methods follows naturally from the requirements for successful analytic continuation into the regions where the relevant sums become classically divergent.

After the (admittedly necessary) length of the last paper, this brings the current one to a blessedly early ending. This leaves the reader to engage in much rejoicing, and to the unalloyed delight of anticipating the next (and final) paper in this series - in which we derive the promised countable family of new results regarding the argument of the zeta function, conditional on RH!

## 2 Proof that $\zeta$ satisfies the generalised root identities for all $\mu \in \mathbb{C} \setminus \{1\}$ when $Re(s_0) > 1$

We fix for the remainder of this paper two real constants $\lambda_0, \mu_0 > 1$. These represent thresholds so that, on the half-planes $Re(s_0) > \lambda_0$ and $Re(\mu) > \mu_0$, the classical convergence we find in certain subsequent formulae will be assured of being uniform.

As usual we let $NT$ denote the set of non-trivial roots of $\zeta$ and $NT_+$ (resp. $NT_-$) denote the subset with positive (resp. negative) imaginary part.

Then, with sums understood to include multiplicities, our starting point (as in [VIII]) is the following explicit formula which is known to hold for $\zeta$ (see e.g. [9], pg 35):

$$-\frac{\zeta'(s)}{\zeta(s)} = \left\{ \begin{array}{c} -\frac{1}{2}\ln\pi + \frac{1}{s-1} + \frac{\Gamma'(\frac{s}{2}+1)}{\Gamma(\frac{s}{2}+1)} \\ \\ -(2s-1)\sum_{\rho\in NT_+} \frac{1}{(s-\rho)(s-(1-\rho))} \end{array} \right\} . \tag{1}$$

This can be viewed either as the Hadamard formula for $\zeta$, or else, as discussed in [VII], as the $\mu = 1$ root identity for $\zeta$ after allowing for a required renormalisation and removal of an obstruction. Differentiating w.r.t. $s$, recalling that $\frac{\mathrm{d}^2}{\mathrm{d}z^2}(\ln(\Gamma(z+1))) = \sum_{n=1}^{\infty} \frac{1}{(z+n)^2}$ and splitting the sum over $\rho$ into separate convergent sums over $NT_+$ and $NT_-$we obtain that

$$\begin{aligned} -\frac{\mathrm{d}^2}{\mathrm{d}s^2}(\ln(\zeta(s)))\Big|_{s=s_0} &= \left\{ \begin{array}{c} -\frac{1}{(s_0-1)^2} + \sum_{n=1}^{\infty} \frac{1}{(s_0+2n)^2} \\ \\ +\sum_{\rho\in NT_+} \frac{1}{(s_0-\rho)^2} + \sum_{\tilde{\rho}\in NT_-} \frac{1}{(s_0-\tilde{\rho})^2} \end{array} \right\} \\ &= \sum_{\{roots\ r_i\ of\ \zeta\}} \frac{M_i}{(s_0-r_i)^2} \quad . \end{aligned} \tag{2}$$

Here the last sum is over the "generalised roots" of $\zeta$, comprising the pole at $s = 1$ (which contributes the first term on the RHS with multiplicity $M_i = -1$), the trivial roots $T$ (all with $M_i = 1$, which contribute the second sum on the RHS), and the non-trivial roots $NT = NT_+ \cup NT_-$ (which contribute the third and fourth sums on the RHS).

Thus the generalised root identity for $\zeta$

$$\frac{-1}{\Gamma(\mu)}\left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^{\mu}(\ln(\zeta(s)))\Big|_{s=s_0} = \mathrm{e}^{i\pi\mu} \sum_{\{roots\ r_i\ of\ \zeta\}} \frac{M_i}{(s_0-r_i)^{\mu}}\ , \quad \mu\in\mathbb{C} \tag{3}$$

is at least certainly true for $\mu = 2$ (and hence also for $\mu \in \mathbb{Z}_{>2}$ by repeated differentiation). As always we shall denote the LHS in (3) by $d_\zeta(s_0, \mu)$ and the

RHS by $r_\zeta(s_0,\mu)$, and we shall refer to these as the derivative and root sides of the root identities for $\zeta$ respectively.

We now prove the following result, which extends this rigorous claim at the discrete values $\mu \in \mathbb{Z}_{\geq 2}$ to arbitrary $\mu$ in the open real interval $(\frac{3}{2}, 2)$:

**Result 1:** *For arbitrary real $\mu \in (\frac{3}{2}, 2)$ the generalised root identity (3) continues to hold. Moreover, for such $\mu$, on the LHS of (3) the derivative side is given by*

$$d_\zeta(s_0,\mu) = -\frac{\mathrm{e}^{i\pi\mu}}{\Gamma(\mu)} \sum_{p\, prime} (\ln p)^\mu \sum_{n=1}^{\infty} n^{\mu-1}\, p^{-ns_0}\,. \tag{4}$$

*Thus, explicitly, for arbitrary real $\mu \in (\frac{3}{2}, 2)$ in (3) we have*

$$d_\zeta(s_0,\mu) = r_\zeta(s_0,\mu) \tag{5}$$

*where*

$$d_\zeta(s_0,\mu) = -\frac{\mathrm{e}^{i\pi\mu}}{\Gamma(\mu)} \sum_{p\, prime} (\ln p)^\mu \sum_{n=1}^{\infty} n^{\mu-1}\, p^{-ns_0}$$

*and*

$$r_\zeta(s_0,\mu) = \mathrm{e}^{i\pi\mu} \sum_{\{roots\, r_i\ of\ \zeta\}} \frac{M_i}{(s_0-r_i)^\mu}\,.$$

**Proof:** For simplicity we restrict initially to $Re(s_0) > \lambda_0$, and $s_0$ real. In [VII] we adopted a Fourier-theoretic definition of $\left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^\mu$ for $\mu \notin \mathbb{Z}_{\geq 1}$. Here it will be more convenient to use the following alternative definition from the theory of fractional calculus.

For $0 < \alpha < 1$, let $J_\alpha$ be the fractional integral operator defined by

$$J_\alpha[f](x) = \frac{\mathrm{e}^{-i\pi\alpha}}{\Gamma(\alpha)} \int_x^\infty (s-x)^{\alpha-1} f(s)\,\mathrm{d}s \quad . \tag{6}$$

Then $J_\alpha$ is clearly well-defined on, for example, continuous functions $f(s)$ on $[\lambda_0,\infty)$ which decay sufficiently fast as $s\to\infty$; and it may be taken as defining $\left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^{-\alpha}$ on such functions for $0<\alpha<1$. For $\frac{3}{2} < \mu < 2$, we can then define $\left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^\mu$ by $J_{2-\mu} \circ \left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^2$.

Since $\lambda_0 > 1$, $f(s) = \frac{1}{(s-r_i)^2}$ is such a decaying, continuous function on $[\lambda_0,\infty)$ for any generalised root, $r_i$, of $\zeta$, and thus in (3) we obtain that for any $\mu \in (\frac{3}{2}, 2)$ we have

$$\begin{aligned} \frac{-1}{\Gamma(\mu)} \left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^\mu (\ln(\zeta(s)))\Big|_{s=s_0} &= \frac{-1}{\Gamma(\mu)} \left(J_{2-\mu} \circ \left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^2\right) [\ln(\zeta(s))]\Big|_{s=s_0} \\ &= \frac{1}{\Gamma(\mu)} J_{2-\mu} \left[\sum_{\{roots\ r_i\ of\ \zeta\}} \frac{M_i}{(s-r_i)^2}\right](s_0)\,. \end{aligned} \tag{7}$$

Now, after changing variables to $u = s - s_0$ in the definition of $J_{2-\mu}$, we have

$$J_{2-\mu}\left[\frac{1}{(s-r_i)^2}\right](s_0) = \frac{e^{i\pi\mu}}{\Gamma(2-\mu)}\int_0^\infty u^{1-\mu}\,\frac{1}{(u+R_i)^2}\,du \tag{8}$$

where $R_i = s_0 - r_i$.

But in [10] we find that

$$\int_0^\infty \frac{x^m}{ax^2+bx+c}\,dx = \frac{\pi\csc(m\pi)}{2^m\delta}\left\{\left(\frac{b+\delta}{a}\right)^m - \left(\frac{b-\delta}{a}\right)^m\right\} \tag{9}$$

for $a, b, c$ real such that $0 < a, c$ and $-2\sqrt{ac} < b$ and $m \in (-1,1)\setminus\{0\}$, where $\delta = \sqrt{b^2-4ac}$.

Since, for $\delta$ small compared with $b$, we have

$$\frac{1}{\delta}\left\{\left(\frac{b+\delta}{a}\right)^m - \left(\frac{b-\delta}{a}\right)^m\right\} = \frac{b^m}{a^m}\left\{\frac{2m}{b} + O(\delta^2)\right\}$$

so in the limit as $\delta \to 0$ the identity (9) becomes

$$\int_0^\infty \frac{x^m}{ax^2+bx+c}\,dx = \frac{m\pi\csc(m\pi)}{2^{m-1}}\cdot\frac{b^{m-1}}{a^m} \tag{10}$$

for $a, b, c$ real such that $0 < a, c$ and $-2\sqrt{ac} < b$ and $m \in (-1,1)\setminus\{0\}$ and $b^2 - 4ac = 0$. In particular

$$\int_0^\infty \frac{x^m}{(x+R)^2}\,dx = m\pi\csc(m\pi)\cdot R^{m-1} \tag{11}$$

for any $R > 0$ real and $m \in (-1,1)\setminus\{0\}$.

Now since the RHS in (11) clearly extends analytically off the positive real axis in $R$ and the integral on the LHS remains uniformly convergent when $R$ acquires an imaginary part, so (11) remains true for arbitrary $R$ in the complex half-plane $Re(R) > 0$; i.e.

$$\int_0^\infty \frac{x^m}{(x+R)^2}\,dx = m\pi\csc(m\pi)\cdot R^{m-1} \tag{12}$$

for arbitrary $Re(R) > 0$ and $m \in (-1,1)\setminus\{0\}$. Moreover it is clear that this holds uniformly in $m$ if, say, we restrict to $Re(R) \geq \lambda_0 - 1$.

It follows in (8) that, since $R_i = (s_0 - r_i)$ has $Re(R_i) \geq \lambda_0 - 1$ for all generalised roots, $r_i$, of $\zeta$, so

$$\begin{aligned} J_{2-\mu}\left[\frac{1}{(s-r_i)^2}\right](s_0) &= \frac{e^{i\pi\mu}}{\Gamma(2-\mu)}\pi(1-\mu)\csc(\pi(1-\mu))\cdot(s_0-r_i)^{-\mu} \\ &= e^{i\pi\mu}\Gamma(\mu)\cdot(s_0-r_i)^{-\mu} \quad . \end{aligned}$$

But since this holds uniformly, so in (7) we may reverse the order of $J_{2-\mu}$ and $\sum_{\{roots\, r_i\, of\, \zeta\}}$ to obtain

$$\frac{-1}{\Gamma(\mu)}\left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^{\mu}(\ln(\zeta(s)))\Big|_{s=s_0} = \mathrm{e}^{i\pi\mu}\sum_{\{roots\, r_i\, of\, \zeta\}}\frac{M_i}{(s_0-r_i)^{\mu}}$$

and this proves the first of the claims in Result 1.

To derive also the remaining claim regarding the explicit form of $d_\zeta(s_0,\mu)$ in equation 4, recall that since $Re(s_0)>\lambda_0$, so $\zeta$ satisfies the Euler product formula (uniformly in $s$):

$$\zeta(s) = \prod_{p\, prime}(1-p^{-s})^{-1} \quad . \tag{13}$$

Thus

$$\ln(\zeta(s)) = \sum_{p\, prime}\left\{p^{-s}+\frac{1}{2}p^{-2s}+\frac{1}{3}p^{-3s}+\ldots\right\} \tag{14}$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}(\ln(\zeta(s))) = \sum_{p\, prime}\sum_{n=1}^{\infty}n(\ln p)^2 p^{-ns} \quad . \tag{15}$$

Now on $Re(s)>\lambda_0$ the function $p^{-ns}$ is clearly uniformly bounded, continuous and sufficiently rapidly decaying to work as above and obtain, for $\mu\in(\frac{3}{2},2)$:

$$\frac{-1}{\Gamma(\mu)}\left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^{\mu}(\ln(\zeta(s)))\Big|_{s=s_0} = -\frac{1}{\Gamma(\mu)}\sum_{p\, prime}\sum_{n=1}^{\infty}n(\ln p)^2 J_{2-\mu}\left[p^{-ns}\right](s_0) \tag{16}$$

But, after again changing variables to $u=s-s_0$ and then to $v=n\ln p\cdot u$, and invoking Euler's well-known integral identity for $\Gamma$, we have that

$$\begin{aligned} J_{2-\mu}\left[p^{-ns}\right](s_0) &= \frac{\mathrm{e}^{i\pi\mu}}{\Gamma(2-\mu)}p^{-ns_0}\int_0^{\infty}u^{1-\mu}\,p^{-nu}\,\mathrm{d}u \\ &= \frac{\mathrm{e}^{i\pi\mu}}{\Gamma(2-\mu)}p^{-ns_0}\int_0^{\infty}\frac{1}{(n\ln p)^{2-\mu}}v^{1-\mu}\,\mathrm{e}^{-v}\,\mathrm{d}v \\ &= \mathrm{e}^{i\pi\mu}(n\ln p)^{\mu-2}\,p^{-ns_0} \quad . \end{aligned} \tag{17}$$

It follows in equation (16) that we obtain

$$\frac{-1}{\Gamma(\mu)}\left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^{\mu}(\ln(\zeta(s)))\Big|_{s=s_0} = -\frac{\mathrm{e}^{i\pi\mu}}{\Gamma(\mu)}\sum_{p\, prime}(\ln p)^{\mu}\sum_{n=1}^{\infty}n^{\mu-1}\,p^{-ns_0}$$

and this verifies formula (4) in Result 1 as promised. Formula (5) then follows simply from combining the two claims just proven - namely the fact that

$r_\zeta(s_0,\mu) = d_\zeta(s_0,\mu)$ on $\mu \in (\frac{3}{2},2)$ and this formula for $d_\zeta(s_0,\mu)$ - and this completes the proof of Result 1.

Having proven Result 1 and in particular identity (5) on $\mu \in (\frac{3}{2},2)$, we now observe that, since we are restricting to $Re(s_0) > \lambda_0$, so on the LHS the function

$$d_\zeta(s_0,\mu) = -\frac{\mathrm{e}^{i\pi\mu}}{\Gamma(\mu)} \sum_{p\,prime} (\ln p)^\mu \sum_{n=1}^{\infty} n^{\mu-1}\, p^{-ns_0} \tag{18}$$

in fact extends immediately to be a well-defined analytic function not just on $Re(\mu) > 1$, but on all $\mu \in \mathbb{C} \setminus \{1\}$.

Moreover, since the factor $\Gamma(\mu)$ in the denominator of $d_\zeta(s_0,\mu)$ is singular at $\mu \in \mathbb{Z}_{\leq 0}$, while the sums in the numerator are classically convergent at such $\mu$, it follows that we have identically as functions of $s_0$ (on $Re(s_0) > \lambda_0$) that

$$d_\zeta(s_0,\mu) = 0 \qquad \forall \mu \in \mathbb{Z}_{\leq 0}\,. \tag{19}$$

On the other hand, for arbitrary $Re(s_0) > \lambda_0$, $r_\zeta(s_0,\mu)$ is clearly itself an analytic function in $\mu$ on the half-plane $Re(\mu) > 1$; and since by Result 1 $d_\zeta(s_0,\mu) = r_\zeta(s_0,\mu)$ on the interval $\mu \in (\frac{3}{2},2)$ which has a limit point, so we obtain the following result by uniqueness of analytic continuation:

**Result 2:** *The function* $r_\zeta(s_0,\mu) = \mathrm{e}^{i\pi\mu} \sum_{\{roots\, r_i\ of\ \zeta\}} \frac{M_i}{(s_0-r_i)^\mu}$, *which is classically convergent to an analytic function of* $\mu$ *on the half-plane* $Re(\mu) > 1$, *has unique analytic continuation to all of* $\mu \in \mathbb{C} \setminus \{1\}$ *given by*

$$r_\zeta(s_0,\mu) = d_\zeta(s_0,\mu) \tag{20}$$

*where* $d_\zeta(s_0,\mu)$ *is as given by formula*(18) *for arbitrary* $\mu \in \mathbb{C} \setminus \{1\}$. *Under this analytic continuation we have, identically as functions of* $s_0$ *(on* $Re(s_0) > \lambda_0$*), that*

$$r_\zeta(s_0,\mu) = 0 \qquad \text{whenever } \mu \in \mathbb{Z}_{\leq 0} \quad . \tag{21}$$

**Comments: (i)** Result 2 proves that $\zeta$ does satisfy the generalised root identities for all $\mu \in \mathbb{C} \setminus \{1\}$ as promised and, moreover, that $d_\zeta(s_0,\mu)$, and hence also $r_\zeta(s_0,\mu)$, is identically zero whenever $\mu \in \mathbb{Z}_{\leq 0}$.

In light of the singularity of $\Gamma(\mu)$ at such $\mu$, it might be thought that this latter fact would always be true whenever any function satisfies the generalised root identities, but this is not the case. For example we saw, when using the Fourier-theoretic approach in [VII] applied to the function $\Gamma(z+1)$, that sometimes the singularity of $\Gamma(\mu)$ at $\mu \in \mathbb{Z}_{\leq 0}$ coincides with the arrival of $\frac{1}{\xi_+}$-pieces in the Fourier integrand to leave $\delta$-function contributions and non-zero outcomes - so that, for example, $d_\Gamma(s_0, 0) = -s_0 - \frac{1}{2}$, $d_\Gamma(s_0,-1) = -\frac{1}{2}s_0^2 - \frac{1}{2}s_0 - \frac{1}{12}$ and so on.

The fact that $d_\zeta(s_0,\mu)$ is identically zero whenever $\mu \in \mathbb{Z}_{\leq 0}$ is thus non-trivial and is central to the derivation, in our next paper, of a family of integral identities for $S(t)$, conditional on RH (just as it was central to our derivation in [VIII] of the first of these identities from this fact for $\mu = -2$).

**(ii)** Note that no assumption of the Riemann hypothesis has been required for our proof that $\zeta$ satisfies the generalised root identities via results 1 and 2 in this section. This will change, however, for our generalised Césaro analysis in the next section.

# 3 Constructive analytic continuation of $r_\zeta(s_0, \mu)$ by generalised Césaro methods

In [VIII] we have already shown in detail how generalised geometric Césaro convergence provides a framework for constructively extending $r_\zeta(s_0, \mu)$ from the half-plane $Re(\mu) > 1$ to all of $\mu \in \mathbb{C} \setminus \{1\}$ whenever $Re(s_0) > 1$.

While we showed that this facilitates evaluation of $r_\zeta(s_0, \mu)$ for all such $\mu$, we did not not fully explain why the resulting extension is the unique *analytic* continuation of $r_\zeta(s_0, \mu)$ beyond its half-plane of classical convergence; and, related to this, why there are no "anomalies" or removable singularities in this Césaro-extension at $\mu \in \mathbb{Z}_{\leq 0}$.

We address these two questions in this section, while avoiding as much as possible duplicating the working already undertaken in [VIII].

**Preliminaries:** To begin, note that for our analysis in this section we do need (as in [VIII]) to assume that the Riemann hypothesis holds:

$$RH : \ All\ non-trivial\ roots\ of\ \zeta\ lie\ on\ the\ critical\ line. \tag{22}$$

We also recall that $r_\zeta(s_0, \mu)$ is a combination of four separate pieces arising from the contributions of the pole at $s = 1$; the trivial roots, $T$; and the two halves of the non-trivial roots, $NT_+$ and $NT_-$:

$$r_\zeta(s_0, \mu) = r_P(s_0, \mu) + r_T(s_0, \mu) + r_{NT_+}(s_0, \mu) + r_{NT_-}(s_0, \mu) \tag{23}$$

where

$$r_P(s_0, \mu) = -\mathrm{e}^{i\pi\mu} \frac{1}{(s_0 - 1)^\mu} \tag{24}$$

$$r_T(s_0, \mu) = \mathrm{e}^{i\pi\mu} \sum_{j=1}^{\infty} \frac{1}{(s_0 + 2j)^\mu} \tag{25}$$

$$r_{NT_+}(s_0, \mu) = \mathrm{e}^{i\pi\mu} \sum_{\rho_i \in NT_+} \frac{M_i}{(s_0 - \rho_i)^\mu} \tag{26}$$

and

$$r_{NT_-}(s_0, \mu) = \mathrm{e}^{i\pi\mu} \sum_{\tilde{\rho}_i \in NT_-} \frac{M_i}{(s_0 - \tilde{\rho}_i)^\mu} \quad . \tag{27}$$

**Analytic continuation of $r_P(s_0,\mu)$ and $r_T(s_0,\mu)$:** Clearly $r_P(s_0,\mu)$ extends immediately to a well-defined, analytic function on all of $\mu \in \mathbb{C}$ (since we are restricting to $Re(s_0) > \lambda_0$).

Next, let us consider the analytic continuation of the trivial-root piece, $r_T(s_0,\mu)$. To show that its extension by generalised geometric Césaro methods provides its correct analytic continuation from the classically convergent half-plane $Re(\mu) > 1$, we m11imic the approach used by Hardy in demonstrating the analytic continuation of $\zeta$ from its defining series $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ in [11, section 13.10, pg 332].

Our starting point is the Euler-McLaurin sum formula, which tells us that

$$\sum_{j=1}^{k} \frac{1}{(s_0+2j)^{\mu}} = \left\{ \begin{array}{c} \frac{1}{2}\frac{1}{1-\mu}(s_0+2k)^{1-\mu} + C_T(s_0,\mu) + \frac{1}{2}(s_0+2k)^{-\mu} \\ \\ -\frac{\mu}{6}(s_0+2k)^{-\mu-1} \\ \\ +\frac{\mu(\mu+1)(\mu+2)}{90}(s_0+2k)^{-\mu-3} + \ldots \end{array} \right\}. \tag{28}$$

On the half-plane $Re(\mu) > 1$ all the terms $(s_0+2k)^{\rho}$ in this expression converge classically to 0 as $k \to \infty$ and since on this half-plane

$$r_T(s_0,\mu) = e^{i\pi\mu} \lim_{k\to\infty} \sum_{j=1}^{k} \frac{1}{(s_0+2j)^{\mu}} \tag{29}$$

so it follows in (28) that $r_T(s_0,\mu) = e^{i\pi\mu} C_T(s_0,\mu)$ on $Re(\mu) > 1$.

But by the uniqueness of analytic continuation, it follows that (up to the factor of $e^{i\pi\mu}$ which we may safely ignore in what follows) $C_T(s_0,\mu)$ as defined in equation (28) is precisely the quantity which provides the correct analytic continuation of $r_T(s_0,\mu)$ to $\mu$-values with $Re(\mu) \le 1$, $\mu \ne 1$. It follows that we have

**Observation 1:** *For any $\mu$ with $Re(\mu) \le 1$, $\mu \ne 1$, the correct analytically continued value of $r_T(s_0,\mu)$ is obtained by applying to the sum in* (28) *any summability method which is regular and prescribes* 0 *limit to (i.e. "ignores") all the terms* $\frac{(s_0+2k)^{1-\mu}}{2(1-\mu)}$, $\frac{1}{2}(s_0+2k)^{-\mu}$, $\frac{\mu}{6}(s_0+2k)^{-\mu-1}$, *etc.*

But this is of course precisely what the generalised geometric Césaro method achieves.

To see this, note that the "geometric" variable for such generalised Césaro calculations is $z = s_0 + 2k + \alpha$, $0 \le \alpha < 2$, since the set of shifted trivial roots of $\zeta$ is the set $\{s_0+2j\}_{j=1}^{\infty}$ lying along the contour $\gamma : t \to s_0 + t$. Powers of this geometric variable, $z^{\rho}$, can then be expanded in descending powers of $(s_0+2k)$, albeit with coefficients containing extra factors of $\alpha$, namely

$$z^{\rho} = (s_0+2k)^{\rho} + \rho(s_0+2k)^{\rho-1}\alpha + \frac{\rho(\rho-1)}{2!}(s_0+2k)^{\rho-2}\alpha^2 + \ldots \tag{30}$$

But we showed in detail in [VIII] that by applying sufficiently high powers of $P$ we may then replace any such expression with a strongly Césaro-equivalent one where the coefficients involving powers of $\alpha$ have been replaced by constant coefficients.

For $\mu \notin \mathbb{Z}_{\leq 0}$, it thus follows immediately that generalised geometric Césaro convergence achieves what is required for correct analytic continuation in observation 1, since we may therefore clearly re-express all the terms $\frac{(s_0+2k)^{1-\mu}}{2(1-\mu)}$, $\frac{1}{2}(s_0+2k)^{-\mu}$, $\frac{\mu}{6}(s_0+2k)^{-\mu-1}$ etc as strongly Césaro-equivalent to linear combinations of pure powers, $z^\rho$ with $\rho \neq 0$, and these immediately all have generalised Césaro limit 0.

It is thus only when $\mu \in \mathbb{Z}_{\leq 0}$ that there is any real question as to the validity of the Césaro approach in correctly analytically continuing $r_T(s_0, \mu)$. At such points we need to ensure that Césaro methods don't lead to anomalous values; and for this we need to ensure that, in the strongly Césaro-equivalent final expansion in terms of $z$ just discussed, we avoid any power $z^\rho$ which has $\rho \to 0$ as $\mu$ approaches the integer value in question.

But we showed in [VIII] (and effectively identically in earlier work in [I] on the Césaro definition of $\zeta$, and in [VII] on the root identities for $\Gamma$) that in fact the linear combinations in powers of $z$ which apply for $\mu$ near such points extend perfectly to the points in question themselves, without acquiring any additional "anomalous" constant terms. Thus the Césaro evaluations there remain consistent with observation 1 and so correct as analytic continuation across these points.

For example, in a neighbourhood of 0 the strong Césaro result which ensures successful evaluation across $\mu = 0$ is that $P[\frac{1}{2}(s_0 + 2k)^{-\mu}(\alpha - 1)] = o(1)$. By equations 28 and 30 this ensures that for all $\mu$ near 0, including $\mu = 0$, we have

$$\begin{aligned}\sum_{j=1}^{k} \frac{1}{(s_0+2j)^\mu} &\overset{C}{\simeq} \frac{1}{2}\frac{1}{1-\mu}(s_0+2k)^{1-\mu} + \frac{1}{2}(s_0+2k)^{-\mu}\alpha + C_T(s_0,\mu) \\ &\overset{C}{\simeq} \frac{1}{2}\frac{1}{1-\mu} z^{1-\mu} + C_T(s_0,\mu) \overset{C}{\sim} C_T(s_0,\mu)\,. \end{aligned} \tag{31}$$

Likewise, in a neighbourhood of $\mu = -1$, we have that $P^2[\frac{1}{2}(s_0+2k)^{-\mu}(\alpha-1) - \frac{\mu}{12}(s_0+2k)^{-\mu-1}(3\alpha^2-2)] = o(1)$; in a neighbourhood of $\mu = -2$ we have that $P^3[\frac{1}{2}(s_0+2k)^{-\mu}(\alpha-1) - \frac{\mu}{12}(s_0+2k)^{-\mu-1}(3\alpha^2-2) + \frac{\mu(\mu+1)}{12}(s_0+2k)^{-\mu-2}\alpha^3] = o(1)$; and so on. In each case these strong Césaro relationships ensure that we continue to have

$$\sum_{j=1}^{k} \frac{1}{(s_0+2j)^\mu} \overset{C}{\simeq} \frac{1}{2}\frac{1}{1-\mu} z^{1-\mu} + C_T(s_0,\mu) \overset{C}{\sim} C_T(s_0,\mu) \tag{32}$$

for all $\mu$ near the integer point, including at the point itself.

Continuing leftwards, one strip at a time, we thus see that our generalised Césaro methods extend successfully across these negative integer points without picking up anomalies, giving us the correct analytic continuation of $r_T(s_0, \mu)$, and in particular the correct values for it at every $\mu \in \mathbb{Z}_{\leq 0}$.

**Analytic continuation of** $r_{NT_+}(s_0,\mu)+r_{NT_-}(s_0,\mu)$**:** Turning now to the non-trivial root pieces, $r_{NT_+}(s_0,\mu)$ and $r_{NT_-}(s_0,\mu)$, the explanation of why their generalised geometric Césaro extension in [VIII] is likewise naturally analytic for $\mu \in \mathbb{C}\setminus\{1\}$ follows the same essential logic.

For $\mu \notin \mathbb{Z}_{\leq 0}$ this extension is analytic for each of the pieces, $r_{NT_+}(s_0,\mu)$ and $r_{NT_-}(s_0,\mu)$, independently. We shall outline the demonstration of this only for the case of $r_{NT_+}(s_0,\mu)$ - the case of $r_{NT_-}(s_0,\mu)$ follows identical reasoning.

For the cases of $\mu \in \mathbb{Z}_{\leq 0}$, however, it is only the sum $r_{NT_+}(s_0,\mu)+r_{NT_-}(s_0,\mu)$ which retains analyticity across these points, owing to the presence of (cancelling) poles in each of the two components independently; and we show how this arises within our generalised Césaro framework, once we have augmented it with one further stipulation.

Since we are assuming RH, the non-trivial roots are all of the form $\rho_i = \frac{1}{2}+i\gamma_i$, $\gamma_i \in \mathbb{R}$ and thus, as in [VIII], we have that the p-sum for $r_{NT_+}(s_0,\mu) = \mathrm{e}^{i\pi\mu}\sum_{\rho_i\in NT_+}\frac{M_i}{(s_0-\rho_i)^\mu}$ is given by

$$\begin{aligned} & \mathrm{e}^{i\pi\mu}\sum_{\{Im(\rho_i)<T\}}\frac{M_i}{(s_0-\rho_i)^\mu} \\ = \quad & \mathrm{e}^{i\frac{3\pi}{2}\mu}\left\{\begin{array}{c} \sum_{\{\gamma_i<T\}} M_i\gamma_i^{-\mu} \\ \\ -i\mu(s_0-\frac{1}{2})\sum_{\{\gamma_i<T\}} M_i\gamma_i^{-\mu-1} \\ \\ -\frac{\mu(\mu+1)(s_0-\frac{1}{2})^2}{2!}\sum_{\{\gamma_i<T\}} M_i\gamma_i^{-\mu-2} \\ \\ +i\frac{\mu(\mu+1)(\mu+2)(s_0-\frac{1}{2})^3}{3!}\sum_{\{\gamma_i<T\}} M_i\gamma_i^{-\mu-3}+\ldots \end{array}\right\} \end{aligned} \tag{33}$$

To replicate the argument we just used for $r_T(s_0,\mu)$ the key is to express this p-sum as an asymptotic expansion in $T$ around $\infty$ in the same fashion we used the Euler-McLaurin formula in the trivial-root case.

We do this, as in [VIII], by considering each sum of the form $\sum_{\{\gamma_i<T\}} M_i\gamma_i^\nu$ as the integral

$$\sum_{\{\gamma_i<T\}} M_i\gamma_i^\nu = \int_0^T t^\nu\,\mathrm{d}N(t) \tag{34}$$

and then using the Riemann-von Mangoldt formula, which gives $N(t)$ as

$$N(T) = \check{N}(T)+S(T)+\frac{1}{\pi}\delta(T) \tag{35}$$

where

$$\check{N}(T) = \frac{T}{2\pi}\ln(\frac{T}{2\pi})-\frac{T}{2\pi}+\frac{7}{8} \tag{36}$$

and $S(T)$ is the famous argument of the zeta function, and

$$\delta(T) = \frac{T}{4}\ln\left(1+\frac{1}{4T^2}\right) + \frac{1}{4}\tan^{-1}\left(\frac{1}{2T}\right) + \frac{T}{2}\int_0^\infty \frac{\overset{\vee}{q}_0(u)}{(u+\frac{1}{4})^2+(\frac{T}{2})^2}\mathrm{d}u\ . \quad (37)$$

Here $\overset{\vee}{q}_0(u)$ is the saw-tooth function (defined in [III]) which rises linearly from $-\frac{1}{2}$ to $\frac{1}{2}$ on each integer interval $[k, k+1)$.

As we saw in [VIII], the resulting asymptotic series is a little more intricate than in the Euler-McLaurin case, but can be expressed for generic $\mu$ as

$$\mathrm{e}^{i\pi\mu}\sum_{Im(\rho_i)<T}\frac{M_i}{(s_0-\rho_i)^\mu} = \mathrm{e}^{i\frac{3\pi}{2}\mu}\left\{\begin{array}{c} divgt(s_0,\mu;T) + oscill(s_0,\mu;T) \\ \\ +C_{NT_+}(s_0,\mu)\end{array}\right\} + o(1) \quad (38)$$

where

$$divgt(s_0,\mu;T) = \left\{\begin{array}{c}\text{A linear combination of terms of the form } T^\lambda \ln T \text{ and} \\ \\ T^\lambda \text{ with descending powers } \lambda = 1-\mu, -\mu, -\mu-1, \ldots\end{array}\right\} \quad (39)$$

and

$$oscill(s_0,\mu;T) = \left\{\begin{array}{c}\text{A linear combination of "oscillatory" terms} \\ \\ \text{rendered convergent by a pure power of } P\end{array}\right\}\ . \quad (40)$$

The terms of the form $T^\lambda \ln T$ arise from integration against the $\check{N}(t)$-component of $N(t)$ in equation 34, as do many of the terms $T^\lambda$ which are pure powers of $T$. The remaining pure powers of $T$ arise from integration against the $\delta(t)$-component of $N(t)$ since it has an asymptotic expansion around $t=\infty$ of the form $\delta(t) = a_1\cdot\frac{1}{t} + a_3\cdot\frac{1}{t^3} + a_5\cdot\frac{1}{t^5} + \ldots$.

The oscillatory terms arise from integration against the $S(t)$-component of $N(t)$ since, as we saw in [VIII], $S(t) = S_0^*(t)$ can be taken as the base of a Césaro-adapted scale $\{S_i^*(t)\}_{i=0}^\infty$ and expressions of the form $t^\lambda S_j^*(t)$ are rendered classically convergent by a sufficiently high power of $P$.

Here, as discussed in [VIII], we are considering in equation 34 only the top-limit of integration - namely $\int^T t^\nu\,\mathrm{d}N(t) = \int^T t^\nu\,\mathrm{d}\check{N}(t) + \frac{1}{\pi}\int^T t^\nu\,\mathrm{d}\delta(t) + \int^T t^\nu\,\mathrm{d}S(t)$ - as contributing to these classically divergent or oscillatory terms as $T\to\infty$. We omit consideration of any contribution from a lower-limit of integration and simply absorb any such finite contributions into the term $C_{NT_+}(s_0,\mu)$.[1]

Now, when $Re(\mu) > 1$, the p-sum is classically convergent and the terms $divgt(s_0,\mu;T)$ and $oscill(s_0,\mu;T)$ in (38) both converge classically to 0 as $T\to$

[1]Doing this also allows us to avoid pointless technicalities regarding integrability near 0 when we split $N(t)$ into its separate pieces ($\check{N}(t)$, $\delta(t)$ and $S(t)$).

$\infty$. Since on this half-plane we have

$$r_{NT_+}(s_0,\mu) = e^{i\pi\mu} \lim_{T\to\infty} \sum_{Im(\rho_i)<T} \frac{M_i}{(s_0-\rho_i)^{\mu}} \tag{41}$$

so it follows in equation (38) that $r_{NT_+}(s_0,\mu) = \mathrm{e}^{i\frac{3\pi}{2}\mu} C_{NT_+}(s_0,\mu)$ on $Re(\mu) > 1$.

But by the uniqueness of analytic continuation it therefore follows that (up to the factor of $\mathrm{e}^{i\frac{3\pi}{2}\mu}$ which we may again safely ignore in what follows) $C_{NT_+}(s_0,\mu)$ as defined in equation (38) is precisely the quantity which provides the correct analytic continuation of $r_{NT_+}(s_0,\mu)$ to $\mu$-values with $Re(\mu) \leq 1$, $\mu \neq 1$. It follows that we have

**Observation 2:** *For any $\mu$ with $Re(\mu) \leq 1$, $\mu \neq 1$, the correct analytically continued value of $r_{NT_+}(s_0,\mu)$ is obtained by applying to the expression in* (38) *any summability method which is regular and prescribes* 0 *limit to (i.e. "ignores") the terms $divgt(s_0,\mu;T)$ and $oscill(s_0,\mu;T)$.*

But, for $\mu \notin \mathbb{Z}_{\leq 0}$, this is again precisely what the generalised geometric Césaro method achieves. This is because, on the one hand, we have $T = i(z - (s_0 - \frac{1}{2}))$ so that all the terms in $divgt(s_0,\mu;T)$ of the form $T^{\lambda}\ln T$ and $T^{\lambda}$ can be re-expressed in terms of expressions of the form $z^{\tau}\ln z$ or $z^{\tau}$ with $\tau \neq 0$, and these are Césaro eigenfunctions or generalised eigenfunctions with generalised Césaro limit 0; while on the other, the terms in $oscill(s_0,\mu;T)$ are all rendered convergent to 0 by a sufficiently high power of $P$.

Thus, as in the case of $r_T(s_0,\mu)$, the only remaining issue is to be sure that the analyticity of the generalised Césaro extension of the *sum*, $r_{NT_+}(s_0,\mu) + r_{NT_-}(s_0,\mu)$, is preserved across the points $\mu \in \mathbb{Z}_{\leq 0}$, with no "anomalies" arising.

For this, as in [VIII], we need to consider the p-sum formulae in equations 38-40 for each of the sums in $r_{NT_+}(s_0,\mu)$ in equation 33 and re-express them all in terms of the rescaled geometric variable $w = \frac{z}{2\pi} = \frac{1}{2\pi}(s_0 - \frac{1}{2} - iT)$; and do the same for the sums arising in $r_{NT_-}(s_0,\mu)$ in terms of the corresponding rescaled geometric variable $\tilde{w} = \frac{\tilde{z}}{2\pi} = \frac{1}{2\pi}(s_0 - \frac{1}{2} + i\tilde{T})$.

When we do this, we saw in [VIII] that for $\mu \in \mathbb{Z}_{\leq 0}$ the asymptotic expansion for the p-sum of $r_{NT_+}(s_0,\mu)$ now acquires a pure log-divergence (i.e. one of the $w^{\lambda}\ln w$ terms now has $\lambda = 0$). Within the generalised geometric Césaro framework, this corresponds to the advent of a simple pole in $\mu$ in $r_{NT_+}(s_0,\mu)$ and we saw this numerically in [VIII] as $\mu \to 0$.

However, the corresponding asymptotic expansion for $r_{NT_-}(s_0,\mu)$ also acquires a pure log-divergence in $\tilde{w}$, signalling the advent of a simple pole in $r_{NT_-}(s_0,\mu)$ at such $\mu \in \mathbb{Z}_{\leq 0}$; and as we saw numerically in [VIII] this pole in $r_{NT_-}(s_0,\mu)$ exactly offsets the pole in $r_{NT_+}(s_0,\mu)$.

In fact, we showed in [VIII] analytically *why* this occurs. For example, at $\mu = 0$, the expression which thus occurs in the combined p-sum of $r_{NT_+}(s_0,\mu) + r_{NT_-}(s_0,\mu)$ is $\ln\left(\frac{\tilde{w}}{w}\right)$. And the same expression $\ln\left(\frac{\tilde{w}}{w}\right)$ arises in analogous fashion at all the other $\mu \in \mathbb{Z}_{\leq 0}$.

It follows that ascribing generalised limit 0 to $\ln\left(\frac{\tilde{w}}{w}\right)$ at $\mu \in \mathbb{Z}_{\leq 0}$ is precisely what is required in order to continue to have $r_{NT_+}(s_0,\mu) + r_{NT_-}(s_0,\mu) = e^{i\pi\mu}\left(C_{NT_+}(s_0,\mu) + C_{NT_-}(s_0,\mu)\right)$ across $\mu \in \mathbb{Z}_{\leq 0}$, and thus guarantee that our extension of $r_{NT}(s_0,\mu)$ remains consistent with Observation 2 and thus also analytic at these points.

The constructive analytic continuation of $r_{NT}(s_0,\mu)$ to all of $\mathbb{C} \setminus \{1\}$ is thus achieved exactly by the generalised geometric Césaro framework, augmented by the simple further stipulation that

$$\underset{w,\tilde{w}\to\infty}{Clim}\ \ln\left(\frac{\tilde{w}}{w}\right) = 0 \quad . \tag{42}$$

This additional stipulation reflects the fact that

$$\ln\left(\frac{\tilde{w}}{w}\right) = \lim_{\mu\to 0}\left(w^{-\mu} - \tilde{w}^{-\mu}\right) \quad . \tag{43}$$

Since $w^{-\mu}$ and $\tilde{w}^{-\mu}$ are both non-trivial Césaro eigefunctions with generalised Césaro limit 0 for all $\mu \neq 0$, it guarantees that within our generalised Césaro framework, our extension of $r_{NT}(s_0,\mu) = r_{NT_+}(s_0,\mu) + r_{NT_-}(s_0,\mu)$ remains analytic across the points $\mu \in \mathbb{Z}_{\leq 0}$ where $\ln\left(\frac{\tilde{w}}{w}\right)$-terms arise, with no associated anomalies.

Overall then, combining the analysis for $r_P(s_0,\mu)$, $r_T(s_0,\mu)$ and $r_{NT}(s_0,\mu) = r_{NT_+}(s_0,\mu) + r_{NT_-}(s_0,\mu)$ we see that the generalised Césaro framework, augmented by the stipulation that $\underset{w,\tilde{w}\to\infty}{Clim}\ \ln\left(\frac{\tilde{w}}{w}\right) = 0$, does indeed provide the correct methodology to constructively extend $r_\zeta(s_0,\mu)$ as an *analytic* function from $Re(\mu) > 1$ to all of $\mathbb{C} \setminus \{1\}$, with no anomalies, as claimed.

**Additional Comment: (i)** Note that, as we remarked in [VIII], the geometric aspects of generalised geometric Césaro convergence become critical at the points $\mu \in \mathbb{Z}_{\leq 0}$. Since $\ln\left(\frac{\tilde{w}}{w}\right) = \ln\left(\frac{\tilde{T}}{T}\right) + i\pi + O\left(\frac{1}{T}\right)$ it follows that we obtain an extra term of $i\pi$ when taking geometric Césaro limits $\underset{w,\tilde{w}\to\infty}{Clim}$, as opposed to just $\underset{T,\tilde{T}\to\infty}{Clim}$ in our expressions for $r_{NT}(s_0,\mu)$ at $\mu \in \mathbb{Z}_{\leq 0}$. As we have seen in [VIII], this is essential in getting the correct evaluations of $r_\zeta(s_0,\mu)$ at such $\mu$.

## 3.1 Final observations regarding the assumption of RH in this section

Throughout the calculations for $r_\zeta(s_0,\mu)$ in this section we have assumed the RH. It is worth spending a moment to isolate precisely where this assumption has been required and to what degree it could be relaxed.

The RH entered into our constructive Césaro calculations of $r_\zeta(s_0,\mu)$ for $Re(\mu) \leq 1$, $\mu \neq 1$ in three ways:

1. It guaranteed that $NT$ roots would lie purely on the contours $\gamma : t \to s_0 - \frac{1}{2} - it$ and $\tilde{\gamma} : \tilde{t} \to s_0 - \frac{1}{2} - i\tilde{t}$ making up the shifted positive and negative components of the critical line. It thus ensured there would be no ambiguity in our integrals defining $r_{NT_\pm}(s_0, \mu)$ and in our attendant geometric Césaro calculations for them;

2. It ensured the absence of any terms arising explicitly from the displacement of any $NT$ roots away from the critical line in equation 33 (and the corresponding formula for $NT_-$). If RH were not assumed, so that we had to write $NT$-roots as $\rho_i = (\frac{1}{2} + \epsilon_i) + i\gamma_i$ with no assumption that the $\epsilon_i$ are all necessarily zero, then equation 33 would become instead

$$\begin{aligned}
&\sum_{\{s_0-NT_+\}} \frac{M_i}{(s_0-\rho_i)^\mu} \\
= \quad &\mathrm{e}^{i\frac{\pi}{2}\mu} \left\{ \begin{array}{c}
\sum_{\{s_0-NT_+\}} M_i \gamma_i^{-\mu} \\
-i\mu(s_0 - \frac{1}{2}) \sum_{\{s_0-NT_+\}} M_i \gamma_i^{-\mu-1} \\
-\frac{\mu(\mu+1)(s_0-\frac{1}{2})^2}{2!} \sum_{\{s_0-NT_+\}} M_i \gamma_i^{-\mu-2} \\
-\frac{\mu(\mu+1)}{2!} \sum_{\{s_0-NT_+\}} M_i \gamma_i^{-\mu-2} \epsilon_i^2 \\
+i\frac{\mu(\mu+1)(\mu+2)(s_0-\frac{1}{2})^3}{3!} \sum_{\{s_0-NT_+\}} M_i \gamma_i^{-\mu-3} \\
+i\frac{\mu(\mu+1)(\mu+2)(s_0-\frac{1}{2})}{2} \sum_{\{s_0-NT_+\}} M_i \gamma_i^{-\mu-3} \epsilon_i^2 + \ldots
\end{array} \right\}
\end{aligned} \tag{44}$$

with a corresponding result for the sum over the $NT$ roots below the real axis, $\sum_{\{s_0-NT_-\}} \frac{M_i}{(s_0-\rho_i)^\mu}$.

In both cases the absence of any information on the behaviour of $\epsilon_i(t)$ as a function of $t$ would render the computation of the sums involving $\epsilon_i$-terms seemingly impossible.

For example, in the case of $\mu = -2$ where we derived in [VIII] an integral identity for $S(t)$ conditional on RH, the additional term $X_\epsilon := \sum_{\{s_0-NT\}} M_i \epsilon_i^2$ would alter this from an explicit integral identity for $S(t)$ to an equation relating this integral to the generalised Césaro value of $X_\epsilon$. Specifically, we would instead have that $-\frac{1}{2} - 2\int_0^\infty t^2 \, \mathrm{d}S(t) + X_\epsilon = 0$.

In this case, it might be thought that we could at least then work backwards from numerical Césaro estimates of $\int_0^\infty t^2 \, \mathrm{d}S(t)$ to get controlling estimates on the size of $X_\epsilon$ and hence some immediate limits on how large any deviations, $\epsilon_i$, off the critical line might be. The problem with such a logical chain lies, unfortunately, in the final step, since we know that Césaro sums can be arbitrarily small or even zero without the summands approaching zero (e.g. $\sum_{j=1}^{\infty} j^2 = \zeta(-2) = 0$), making it impossible to draw inferences regarding $\epsilon_i$-values from upper bounds on $X_\epsilon$!

3. It was necessary for the conditional estimates (see [VIII], quoted from [12]) which guarantee that we may form a Césaro-adapted scale of antiderivatives $S_n^*(T)$ starting from $S_0^*(T) := S(T)$. This is critical in ensuring that we can always handle the $oscill(s_0, \mu; T)$ terms, which arise from the $S(T)$-component of $N(T)$, within the generalised Césaro framework.

These are the three things which could fail in applying our generalised geometric Césaro calculational approach if we were to drop our assumption of RH.

Regarding the first, we think it is possible that the mirror-symmetry of $NT$ roots in the critical line might well make it permissible to continue to treat any such root-pairs off the critical line as nonetheless lying on the contours $\gamma$ and $\tilde{\gamma}$ for purposes of Césaro calculation. Indeed, this seems to us quite likely. The fact that in [VIII] we get the correct value for $r_{NT_+}(s_0, 0) + r_{NT_-}(s_0, 0)$ and hence $r_\zeta(s_0, 0)$ when $\mu = 0$ (where points 2 and 3 above do not in fact apply) lends credence to this possibility.

As regards points 2 and 3, however, at this stage these seem to us insurmountable in general - hence the necessity of assuming RH in this paper and in [VIII]. Some progress might be possible in relaxing condition (2) if we were to assume instead that there are only finitely many NT-roots off the critical line. But even there the difficulties arising from point 3 would still apply, since $\mu = 0$ is the only non-positive integer where we have *unconditional* estimates (on $S(t)$ and $S_1(t)$) which allow us to Césaro-control the oscillatory-divergent terms in $r_{NT_\pm}(s_0, 0)$ and obtain a correspondingly unconditional Césaro limit for these terms.

As a parting observation we note that of course, even though proving it would seem challenging at the moment, it remains possible that Césaro calculations may correctly extend $r_\zeta(s_0, \mu)$ independent of any assumption of RH. In support of this possibility is the fact that, as just noted, the $\mu = 0$ calculation is correct even though, among $\mu \in \mathbb{Z}_{\leq 0}$, this case is the only one where none of the three points listed above is in fact required. Likewise, some support may be gleaned from the fact that the RH-independent numerical computations performed in [VIII] for real $\mu \in (-1, 1)$ showed good agreement with $d_\zeta(s_0, \mu)$, including in the limit as $\mu \to -1$ (where mirror-symmetry of roots in the critical line renders point 2 moot, and where the $S(t)$-related contributions would be expected to cancel, albeit that we do not have an unconditional estimate guaranteeing Césaro-convergence of each of these offsetting terms).

For now, however, an assumption of RH remains critical to the rigour of our working in this paper and in [VIII], and we will continue to make this assumption in our next and final paper in this series. In that paper we will derive our oft-promised family of integral identities for $S(t)$ conditional on RH, extending the one for $\int_0^\infty t^2 \, \mathrm{d}S(t)$ derived at the end of [VIII]. Let the ceaseless clamour of the common man for this next instalment commence!

## 4 Acknowledgements

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