Root Identities II: Root identities for ζ - Part C

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Abstract

We consider the generalised root identities for ζ in detail for $\mu \in \mathbb{Z}_{\leq 0}$ and derive from them, conditional on the Riemann hypothesis (RH), a family of strong Césaro integral identities evaluating $\int_0^\infty t^{2k}\,\mathrm{d}S(t)$ for each $k \in \mathbb{Z}_{\geq 0}$, where $S(t)$ is the argument of the Riemann zeta function. These identities can be interpreted instead as strong Césaro identities for the iterated integrals of $S(t)$ or as evaluations of the Mellin transform of $S'(t)$. In turn, they permit strong Césaro evaluation of integrals of a whole class of general functions against the measure $dS(t)$.

1 Introduction

In [VIII] we gave convincing evidence that $\zeta(s)$ satisfies the generalised root identities for all $\mu \in \mathbb{C}\backslash\{1\}$ and $Re(s_0) > 1$[1], and we also showed that $d_\zeta(s_0, \mu) \equiv 0$ for all $\mu \in \mathbb{Z}_{\leq 0}$. By considering the generalised geometric Césaro evaluation of the root side, $r_\zeta(s_0, \mu)$, we showed that under an assumption of the Riemann hypothesis (RH), the truth of the root identities at such $\mu \in \mathbb{Z}_{\leq 0}$ would in turn imply new results regarding the argument ofthe zeta function, $S(t)$. In particular, we showed that the their validity at $\mu = -2$ would imply that $S(t)$ satisfies $\int_0^\infty t^2\,\mathrm{d}S(t) = -\frac{1}{4}$, where this integral is understood in a generalised Césaro sense as the *strong* Césaro limit $\underset{T\to\infty}{Clim} \int_0^T t^2\,\mathrm{d}S(t)$ via the pure power, P^3, of the Césaro averaging operator.[2]

In [IX] we placed these observations on a solid foundation. We proved that for arbitrary $Re(s_0) > 1$ ζ *does* satisfy the generalised root identities as claimed; and we also proved that the analytic continuation of the root-side, $r_\zeta(s_0, \mu)$, in these identities from the half-plane $Re(\mu) > 1$ to all of $\mu \in \mathbb{C} \setminus \{1\}$ is achieved by the generalised geometric Césaro framework. In so doing, we converted the integral identity above for $\int_0^\infty t^2\,\mathrm{d}S(t)$ from being a conjectural result conditional on ζ satisfying the generalised root identities, to being an established result, evaluating $\int_0^\infty t^2\,\mathrm{d}S(t)$ conditional only on RH.

[1] There is a minor obstruction at $\mu = 1$ but this is purely technical and well-understood

[2] We also validated the generalised root identities for ζ at $\mu = 0$ and $\mu = -1$ under RH, but neither of these cases imposed further conditions on $S(t)$.

In this paper we now extend these calculations for $\mu = 0, -1$ and -2 to arbitrary $\mu \in \mathbb{Z}_{\leq 0}$ under RH.

We show that when $\mu \in \mathbb{Z}_{\leq 0}$ is odd, the generalised root identities are satisfied without imposing any further integral conditions on $S(t)$ - because the symmetry of non-trivial roots above and below the real axis causes the automatic cancellation of the Césaro-terms involving $S(t)$ in the evaluation of $r_{NT}(s_0, \mu)$.

When $\mu \in \mathbb{Z}_{\leq 0}$ is even, however, we get reinforcement rather than cancellation, and we deduce from the generalised root identities at these μ the following family of integral identities for $S(t)$:

$$\int_0^\infty t^{2k} \, \mathrm{d}S(t) \;=\; \frac{(-1)^k}{2^{2k}} \quad \text{for all} \quad k \in \mathbb{Z}_{\leq 0} \tag{1}$$

where the integral, as above, is understood in a generalised geometric Césaro sense and in fact represents the *strong* Césaro limit $\underset{T\to\infty}{\mathcal{C}lim} \int_0^T t^{2k} \, \mathrm{d}S(t)$, via the pure power P^{2k+1}.

The calculations proceed as follows. In section 2.1 we recall the form of the generalised root identities at $\mu \in \mathbb{Z}_{\leq 0}$ and in sections 2.2 and 2.3 we then evaluate the contributions of the pole and the trivial roots ($r_P(s_0, \mu)$ and $r_T(s_0, \mu)$) to the root-side $r_\zeta(s_0, \mu)$ at such $\mu = -n$, $n \in \mathbb{Z}_{\geq 0}$.

In section 2.4 we then turn to the final remaining contribution on the root side - namely from the non-trivial roots, $r_{NT}(s_0, -n)$. Under our assumption of RH, these are all of the form $\rho_i = \frac{1}{2} + i\gamma_i$, occurring in conjugate pairs; and we start by deriving an expansion for $r_{NT}(s_0, -n)$ in terms of expressions of the form $\sum_{\{s_0 - NT\}} M_i \gamma_i^j$, $0 \leq j \leq n$. We then turn in section 3 to the calculation of a general formula for $\sum_{\{s_0 - NT\}} M_i \gamma_i^n$ for arbitrary $n \geq 0$.

Recalling the notation introduced in [VIII] and [IX], we first derive a formula for $\sum_{\{s_0 - NT\}} M_i \gamma_i^n$ in terms of $\overset{\circ}{A}_{-n}$, $\overset{\circ}{B}_{-n}$ and $\overset{\circ}{C}_{-n}$ where these are the contributions from the three components - $\check{N}(T)$, $S(T))$ and $\frac{1}{\pi}\delta(T)$ - in the breakdown of the counting function for the non-trivial roots as $N(T) = \check{N}(T) + S(T) + \frac{1}{\pi}\delta(T)$ under the Riemann-von Mangoldt formula.

In section 3.1 we then derive a formula for $\overset{\circ}{A}_{-n}$ and in section 3.2 we derive the corresponding formula for $\overset{\circ}{C}_{-n}$.

In this latter case, the calculation becomes quite intricate. It entails relating $\int_0^T t^n \, \mathrm{d}\delta(t)$ to the asymptotic expansions of the iterated integrals of $\delta(T)$ and in turn employing a trick for simplifying these by working modulo a certain class of functions before taking Césaro limits. In subsection 3.2.1 we perform this calculation for n odd, and in subsection 3.2.2 we perform it for n even. In both cases we see the true value of wholeheartedly embracing a generalised Césaro perspective in the final step of the calculations, where doing so allows us to make a rapid evaluation of integrals which would otherwise take much greater effort to calculate by traditional means.

In section 3.3 we lastly derive the corresponding formula for $\overset{\circ}{B}_{-n}$. We do this by relating $\int_0^T t^n \, \mathrm{d}S(t)$ to iterated integrals of $S(T)$ and harnessing the fact, as outlined in [VIII], that under RH $S(T)$ forms the base of a Césaro-adapted scale. Hence, in section 3.4, by combining our expressions for $\mathring{A}_{-n}$, $\mathring{B}_{-n}$ and $\mathring{C}_{-n}$, we obtain the desired formula for $\sum_{\{s_0-NT\}} M_i\gamma_i^n$ for arbitrary $n \geq 0$.

In section 4 we then apply this in our formula from section 2.4 to obtain a general expression for $r_{NT}(s_0, -n)$, and we combine this with our formulae for $r_P(s_0, -n)$ and $r_T(s_0, -n)$ to obtain the general expression for $r_\zeta(s_0, -n)$. From this we then consider the implications of the generalised root identities at all $\mu = -n$, $n \in \mathbb{Z}_{\geq 0}$, taking the cases of n odd and n even in turn.

In section 4.1 we show that for n odd the identities are satisfied immediately, without any further implications.

In section 4.2, however, we show that for $n = 2k$ even, the fact that ζ satisfies the generalised root identities at $\mu = -2k$ implies that the argument of the zeta function, $S(t)$, must satisfy the generalised Césaro integral identity given in equation 1 as a strong Césaro result.

We conclude section 4 by making a number of observations regarding this family of results. In section 4.3 we consider the analytic continuation of $r_\zeta(s_0, \mu)$ in s_0 leftwards from $Re(s_0) > 1$ to $s_0 = \frac{1}{2}$ and beyond; and we give a technical reformulation of our results which changes the domain of integration from $t \in [0, \infty)$ (with its natural connection to Mellin transforms) to $t \in (-\infty, \infty)$ (with its possible connection to Fourier and other transforms).

In section 4.4 we then consider the extension of such integrals to the case of integration of a general function containing an additional parameter against the measure $dS(t)$. This connects directly to the theory of Césaro arrays we developed in [IV]-[VI], and this allows us to develop a general formula for such integrals. We give examples of such prospective generalised Césaro calculations and discuss their interpretation in terms of asymptotic series in cases where such interpretation seems otherwise to become problematic.

Finally, in section 5, we discuss briefly some of the significant areas left to explore regarding generalised root identities - including applying generalised Césaro methods to the *derivative* side of these identities for ζ; exploring further the connection to Césaro arrays; and investigating the application of these identities to other functions beyond the cases of ζ, Γ, $\cos(\frac{\pi z}{2})$ and polynomials which have been our primary focus so far. We will (eventually) take up these issues in a future set of papers.

2 The generalised root identities for ζ at $\mu \in \mathbb{Z}_{\leq 0}$ when $Re(s_0) > 1$

2.1 The form of the generalised root identities for ζ at $\mu \in \mathbb{Z}_{\leq 0}$, $Re(s_0) > 1$

We showed in [VIII] and [IX] that at $\mu \in \mathbb{Z}_{\leq 0}$ the derivative side of the generalised root identities is identically zero (for $Re(s_0) > 1$), i.e. $d_\zeta(s_0, -n) = 0$ for all $n \in \mathbb{Z}_{\geq 0}$. Since, by [IX], ζ does satisfy these identities at all $\mu \in \mathbb{C} \setminus \{1\}$, it follows that we have

$$r_\zeta(s_0,-n) = r_P(s_0,-n) + r_T(s_0,-n) + r_{NT}(s_0,-n) = 0 \tag{2}$$

for all $n \in \mathbb{Z}_{\geq 0}$, where r_P, r_T and r_{NT} represent the contributions to r_ζ from the simple pole at $\mu = 1$, the trivial roots T, and the non-trivial roots NT respectively.

We now turn to calculating each of these pieces in turn. In doing so and for the rest of the paper we shall fully embrace the generalised Césaro framework (while also working more succinctly than in previous papers). We do so because an ancillary aim of this paper is to demonstrate how useful and efficient generalised Césaro methodology (and a number of other methods unique to this paper) can be in simplifying complex calculations.

2.2 An expression for $r_P(s_0,-n)$

Since the pole at $\mu = -1$ is a generalised root with multiplicity -1, it follows immediately that we have

$$r_P(s_0,-n) = \mathrm{e}^{-i\pi n} \cdot \frac{-1}{(s_0-1)^{-n}} = (-1)^{n+1}(s_0-1)^n\,. \tag{3}$$

2.3 An expression for $r_T(s_0,-n)$

When $s_0 = 0$ dilation-invariance implies that for the trivial roots $r_j = -2j$, $j \in \mathbb{Z}_{>0}$, the Césaro sum $\sum_{r_i \in \{s_0 - T\}} (s_0 - r_i)^n$ defining $r_T(s_0,-n)$ is given by $2^n\zeta(-n)$.

Defining the family of polynomials $b_{n+1}(k) := \sum_{j=1}^{k} j^n$ as we have done previously[3], and applying dilation invariance once more to the picture for $\sum_{r_i \in \{s_0-T\}}(s_0 - r_i)^n$, it then follows immediately that for generic s_0 we have

$$\begin{aligned} r_T(s_0,-n) &= (-1)^n \cdot \left\{ 2^n\zeta(-n) - 2^n b_{n+1}(\frac{s_0}{2}) \right\} \\ &= (-1)^{n+1}2^n \cdot \left\{ b_{n+1}(\frac{s_0}{2}) - \zeta(-n) \right\}\,. \end{aligned} \tag{4}$$

Since it is well-known that $b_{n+1}(k) - \zeta(-n) = \frac{1}{(n+1)}k^{n+1} + \frac{1}{2}k^n - \sum_{j=1}^{n}\binom{n}{j}\zeta(-j)k^{n-j}$ this can be expressed explicitly as

$$r_T(s_0,-n) = (-1)^{n+1} \cdot \left\{ \frac{1}{2(n+1)} s_0^{n+1} + \frac{1}{2}s_0^n - \sum_{j=1}^{n} \binom{n}{j} \zeta(-j)\, 2^j\, s_0^{n-j} \right\}\,. \tag{5}$$

For example $r_T(s_0,0) = -\frac{1}{2}s_0 - \frac{1}{2}$, $r_T(s_0,-1) = \frac{1}{4}s_0^2 + \frac{1}{2}s_0 + \frac{1}{6}$, $r_T(s_0,-2) = -\frac{1}{6}s_0^3 - \frac{1}{2}s_0^2 - \frac{1}{3}s_0$ and so on.

[3] so $b_1(k) = k$; $b_2(k) = \frac{1}{2}k^2 + \frac{1}{2}k$; $b_3(k) = \frac{1}{3}k^3 + \frac{1}{2}k^2 + \frac{1}{6}k$ and so on

2.4 An initial expression for $r_{NT}(s_0, -n)$

Turning now to $r_{NT}(s_0, -n)$, recall that we are assuming the Riemann hypothesis (RH), so that the non-trivial roots of ζ are all of the form $\rho_i = \frac{1}{2} + i\gamma_i$, $\gamma_i \in \mathbb{R}$. It follows that we have

$$r_{NT}(s_0, -n) = (-1)^n \cdot \sum_{\{s_0 - NT\}} M_i (s_0 - \rho_i)^n = (-1)^n \cdot \sum_{\{s_0 - NT\}} M_i (s_0 - \frac{1}{2} - i\gamma_i)^n \tag{6}$$

and since n is an integer, this last sum may be expanded as

$$r_{NT}(s_0, -n) = (-1)^n \left\{ \begin{array}{c} (s_0 - \frac{1}{2})^n \cdot \sum_{\{s_0 - NT\}} M_i \\ -i\binom{n}{1}(s_0 - \frac{1}{2})^{n-1} \cdot \sum_{\{s_0 - NT\}} M_i \gamma_i \\ -\binom{n}{2}(s_0 - \frac{1}{2})^{n-2} \cdot \sum_{\{s_0 - NT\}} M_i \gamma_i^2 \\ + \dots \\ +(-i)^{n-1}\binom{n}{n-1}(s_0 - \frac{1}{2})^1 \cdot \sum_{\{s_0 - NT\}} M_i \gamma_i^{n-1} \\ +(-i)^n \binom{n}{n} \cdot \sum_{\{s_0 - NT\}} M_i \gamma_i^n \end{array} \right\} \tag{7}$$

It follows that finding a general expression for $r_{NT}(s_0, -n)$ is equivalent to finding a general formula for $\sum_{\{s_0 - NT\}} M_i \gamma_i^j$, $0 \le j \le n$. Indeed this equivalence can be expressed neatly in operator-combinatorial terms by inverting equation 7 to write $\sum_{\{s_0 - NT\}} M_i \gamma_i^n$ in terms of $r_{NT}(s_0, -j)$, $0 \le j \le n$, namely as

$$\begin{aligned} \sum_{\{s_0 - NT\}} M_i \gamma_i^n &= (-i)^n \left\{ \begin{array}{c} (s_0 - \frac{1}{2})^n \cdot r_{NT}(s_0, 0) \\ +\binom{n}{1}(s_0 - \frac{1}{2})^{n-1} \cdot r_{NT}(s_0, -1) \\ + \dots \\ +\binom{n}{n} \cdot r_{NT}(s_0, -n) \end{array} \right\} \\ &= (-i)^n \left\{ \left((s_0 - \frac{1}{2}) + T_\mu^{-1} \right)^n [r_{NT}(s_0, \mu)] \,|_{\mu=0} \right\} \end{aligned} \tag{8}$$

where T_μ is the operator on expressions $r_{NT}(s_0, \mu)$ given by $T_\mu [r_{NT}(s_0, \mu)] \,|_{\mu=\mu_0} = r_{NT}(s_0, \mu_0 + 1)$.

Our focus, however, is on carrying out our overall calculation as directly as possible rather than on pursuing this subsidiary angle. As such we defer any further exploration of this here and instead turn now to finding a general formula for $\sum_{\{s_0 - NT\}} M_i \gamma_i^n$ for arbitrary $n \in \mathbb{Z}_{\ge 0}$.

3 A general formula for $\sum_{\{s_0-NT\}} M_i\gamma_i^n$ and $r_{NT}(s_0,-n)$

Since the non-trivial roots of ζ occur in conjugate pairs, the counting function $N(T)$ - which counts the number of non-trivial roots, ρ_i, with imaginary part satisfying $0<\gamma_i<T$ for $T>0$ - extends naturally to $T<0$ as an odd-function in T. We extend this oddness in turn down to each of the three components - $\check{N}$, S and δ - which make up $N(T)$ in the Riemann-von Mangoldt formula:

$$N(T)=\check{N}(T)+S(T)+\frac{1}{\pi}\delta(T) \quad . \tag{9}$$

Here, as in [VIII] and [IX],

$$\check{N}(T)=\frac{T}{2\pi}\ln(\frac{T}{2\pi})-\frac{T}{2\pi}+\frac{7}{8} \tag{10}$$

and $S(T)$ is the famous argument of the zeta function, and

$$\delta(T)=\frac{T}{4}\ln\left(1+\frac{1}{4T^2}\right)+\frac{1}{4}\tan^{-1}\left(\frac{1}{2T}\right)+\frac{T}{2}\int_0^\infty\frac{\overset{\vee}{q}_0(u)}{(u+\frac{1}{4})^2+(\frac{T}{2})^2}\mathrm{d}u \tag{11}$$

where $\overset{\vee}{q}_0(u)$ in this last integral definition is the saw-tooth function which rises linearly from $-\frac{1}{2}$ to $\frac{1}{2}$ on each integer interval $[k,k+1)$, and which forms (per [III]) the base in a Césaro-adapted scale of functions $\{\overset{\vee}{q}_j(u)\}_{j=0}^{\infty}$.

As in [VIII] and [IX] it is easy to see that the two p-sums we are concerned with, $\sum_{0<\gamma_i<T} M_i\gamma_i^n$ and $\sum_{-\tilde{T}<\gamma_i<0} M_i\gamma_i^n$, can both be expressed as integrals, namely $\int_0^T t^n\mathrm{d}N(t)$ and $\int_{-\tilde{T}}^0 t^n\mathrm{d}N(t)$. But the oddness of $N(T)$ and its constituents means that

$$\int_{-\tilde{T}}^0 t^n\mathrm{d}N(t) \;=\; (-1)^n\int_0^{\tilde{T}}\tilde{t}^n\mathrm{d}N(\tilde{t}) \tag{12}$$

and similarly for the breakdown of this relationship into component pieces. It follows that we have

$$\begin{aligned}\sum_{\{s_0-NT\}} M_i\gamma_i^n &= \underset{z,\tilde{z}\to\infty}{\mathit{Clim}}\left\{\int_0^T t^n\mathrm{d}N(t)+(-1)^n\int_0^{\tilde{T}}\tilde{t}^n\mathrm{d}N(\tilde{t})\right\} &(13)\\ &= \overset{\circ}{A}_{-n}+\overset{\circ}{B}_{-n}+\overset{\circ}{C}_{-n} &(14)\end{aligned}$$

where

$$\overset{\circ}{A}_{-n}=\underset{z,\tilde{z}\to\infty}{\mathit{Clim}}\left\{\int_0^T t^n\mathrm{d}\check{N}(t)+(-1)^n\int_0^{\tilde{T}}\tilde{t}^n\mathrm{d}\check{N}(\tilde{t})\right\} \tag{15}$$

and

$$\overset{\circ}{B}_{-n}=\underset{z,\tilde{z}\to\infty}{\mathit{Clim}}\left\{\int_0^T t^n\mathrm{d}S(t)+(-1)^n\int_0^{\tilde{T}}\tilde{t}^n\mathrm{d}S(\tilde{t})\right\} \tag{16}$$

and

$$\mathring{C}_{-n} = \frac{1}{\pi} \cdot \underset{z,\tilde{z}\to\infty}{Clim} \left\{ \int_0^T t^n \mathrm{d}\delta(t) + (-1)^n \int_0^{\tilde{T}} \tilde{t}^n \mathrm{d}\delta(\tilde{t}) \right\} . \tag{17}$$

Note here, of course, that while the above formulae for the p-sums are agnostic to geometry, the formulae in equations 15-17 are *geometric* generalised Césaro limits, meaning that any resulting expressions in T and $\tilde{T}$ need to be interpreted in a generalised Césaro sense having regard to the associated geometric variables $z = (s_0 - \frac{1}{2}) - iT$ and $\tilde{z} = (s_0 - \frac{1}{2}) + i\tilde{T}$ (or dilated variants of these variables).

We turn now to the detailed computation of expressions for each of these three components, $\mathring{A}_{-n}$, $\mathring{B}_{-n}$ and $\mathring{C}_{-n}$, in turn. To facilitate this, we recall the following notational preliminaries introduced in [VIII], which simplify all our calculations and which apply in everything that follows.

Notation: For a given T and $\tilde{T}$, in addition to the direct geometric variables z and $\tilde{z}$ defined above, we denote by w and $\tilde{w}$ the scaled geometric variables given by

$$w = \frac{z}{2\pi} \qquad (resp. \quad \tilde{w} = \frac{\tilde{z}}{2\pi}) \tag{18}$$

and by u and $\tilde{u}$ the corresponding scaled parameters

$$u = \frac{T}{2\pi} \qquad (resp. \quad \tilde{u} = \frac{\tilde{T}}{2\pi}) . \tag{19}$$

Letting

$$q = \frac{(s_0 - \frac{1}{2})}{2\pi} \tag{20}$$

we then have

$$u = iw \cdot \left(1 - \frac{q}{w}\right) \qquad (resp. \quad \tilde{u} = -i\tilde{w} \cdot \left(1 - \frac{q}{\tilde{w}}\right)) \quad . \tag{21}$$

Thus, for example, we have that

$$\ln u = i\frac{\pi}{2} + \ln w - \frac{q}{w} - \frac{1}{2}\frac{q^2}{w^2} - \dots \qquad (resp. \quad \ln \tilde{u} = -i\frac{\pi}{2} + \ln \tilde{w} - \frac{q}{\tilde{w}} - \frac{1}{2}\frac{q^2}{\tilde{w}^2} - \dots) \tag{22}$$

while dilation-invariance also means that $\underset{z,\tilde{z}\to\infty}{Clim}$ and $\underset{w,\tilde{w}\to\infty}{Clim}$ are interchangeable.

As in [VIII] we also define a ladder of integrals of $N(t)$ by

$$N_0(T) := N(T), \quad \text{and} \quad N_i(T) := \int_0^T N_{i-1}(t)\mathrm{d}t \quad \forall i \in \mathbb{Z}_{>0} \tag{23}$$

and we similarly define corresponding ladders for each of the components $\check{N}_i(T)$, $\delta_i(T)$ and $S_i(T)$ for all $i \in \mathbb{Z}_{\geq 0}$. Note that the lower limit in all of the integrals in these ladders is 0.

3.1 A general formula for $\overset{\circ}{A}_{-n}$

Letting $v = \frac{t}{2\pi}$ we have $\mathrm{d}\check{N}(t) = \ln v \, \mathrm{d}v$, and so

$$\begin{aligned} \int_0^T t^n \mathrm{d}\check{N}(t) &= (2\pi)^n \cdot \int_0^u v^n \ln v \, \mathrm{d}v \\ &= (2\pi)^n \cdot \left\{ \frac{1}{(n+1)} u^{n+1} \ln u - \frac{1}{(n+1)^2} u^{n+1} \right\} \end{aligned} \tag{24}$$

and similarly for $\int_0^{\tilde{T}} \tilde{t}^n \mathrm{d}\check{N}(\tilde{t})$. Now, within the geometric generalised Césaro framework we have immediately that $\underset{w\to\infty}{Clim}\ w^j = 0$, and so it follows from equation 21 that

$$u^{n+1} \overset{C}{\sim} i^{n+1}(-1)^{n+1} q^{n+1} \tag{25}$$

while equation 22 combined with equation 21leads in similar fashion to the result that

$$\begin{aligned} u^{n+1} \ln u &\overset{C}{\to} \underset{w\to\infty}{Clim}\ i^{n+1} w^{n+1} \cdot \left\{ \sum_{j=0}^{n+1} (-1)^j \binom{n+1}{j} \frac{q^j}{w^j} \right\} \\ &\qquad \cdot \left\{ i\frac{\pi}{2} + \ln w - \frac{q}{w} - \frac{1}{2}\frac{q^2}{w^2} - \ldots \right\} \\ &= \underset{w\to\infty}{Clim}\ i^{n+1} \left\{ \begin{array}{c} (-1)^{n+1} q^{n+1} \ln w + i\frac{\pi}{2}(-1)^{n+1} q^{n+1} \\ \\ + \sum_{j=0}^{n} (-1)^{j+1} \frac{\binom{n+1}{j}}{(n+1-j)} q^{n+1} \end{array} \right\}. \end{aligned} \tag{26}$$

In identical fashion, via the corresponding formulae for $\tilde{u}$ and $\tilde{w}$, we have that

$$\tilde{u}^{n+1} \overset{C}{\sim} (-i)^{n+1}(-1)^{n+1} q^{n+1} \tag{27}$$

and

$$\tilde{u}^{n+1} \ln \tilde{u} \overset{C}{\to} \underset{\tilde{w}\to\infty}{Clim}\ (-i)^{n+1} \left\{ \begin{array}{c} (-1)^{n+1} q^{n+1} \ln \tilde{w} - i\frac{\pi}{2}(-1)^{n+1} q^{n+1} \\ \\ + \sum_{j=0}^{n} (-1)^{j+1} \frac{\binom{n+1}{j}}{(n+1-j)} q^{n+1} \end{array} \right\}. \tag{28}$$

Thus, in equation 24 we get that

$$\int_0^T t^n \mathrm{d}\check{N}(t) \overset{C}{\sim} \frac{(i\cdot(s_0 - \frac{1}{2}))^{n+1}}{2\pi} \cdot \left\{ \begin{array}{c} \frac{1}{(n+1)} \left[\begin{array}{c} (-1)^{n+1} \ln w + (-1)^{n+1}\, i\frac{\pi}{2} \\ \\ + \sum_{j=0}^{n} (-1)^{j+1} \frac{\binom{n+1}{j}}{(n+1-j)} \end{array} \right] \\ - \frac{1}{(n+1)^2} (-1)^{n+1} \end{array} \right\} \tag{29}$$

and from the corresponding formula for $\int_0^{\tilde{T}} \tilde{t}^n \mathrm{d}\check{N}(\tilde{t})$ we get that

$$\int_0^{\tilde{T}} \tilde{t}^n \mathrm{d}\check{N}(\tilde{t}) \overset{C}{\sim} \frac{(-i\cdot(s_0-\frac{1}{2}))^{n+1}}{2\pi} \cdot \left\{ \begin{array}{c} \frac{1}{(n+1)} \left[\begin{array}{c} (-1)^{n+1} \ln \tilde{w} - (-1)^{n+1}\, i\frac{\pi}{2} \\ \\ +\sum_{j=0}^{n} (-1)^{j+1} \frac{\binom{n+1}{j}}{(n+1-j)} \end{array} \right] \\ \\ -\frac{1}{(n+1)^2}(-1)^{n+1} \end{array} \right\} \tag{30}$$

Combining these formulae in equation 15 we find, after cancellation of all but the first two terms in each of these expressions, that

$$\overset{\circ}{A}_{-n} \overset{C}{\sim} \frac{1}{2\pi}\, i^{n+1}\, (s_0-\frac{1}{2})^{n+1} \cdot \left\{ \frac{(-1)^{n+1}}{(n+1)} \ln\left(\frac{w}{\tilde{w}}\right) + \frac{(-1)^{n+1}}{(n+1)} \cdot i\pi \right\} . \tag{31}$$

But, as discussed at length in [VIII] and [IX], recall that we also have $\underset{w,\tilde{w}\to\infty}{Clim} \ln\left(\frac{w}{\tilde{w}}\right) = 0$; this stipulation reflects the cancellation of simple poles arising from the non-trivial roots above and below the real axis and ensures analyticity of our generalised Césaro extension. It follows finally that we have

$$\overset{\circ}{A}_{-n} = \frac{(-i)^n}{2(n+1)}\, (s_0-\frac{1}{2})^{n+1} . \tag{32}$$

3.2 A general formula for $\overset{\circ}{C}_{-n}$

Turning next to $\overset{\circ}{C}_{-n}$, recall from its definition in equation 11 that as $T\to\infty$, $\delta(T)$ has an asymptotic expansion in decaying odd powers of T of the form

$$\delta(T) = a_1\frac{1}{T} + a_3\frac{1}{T^3} + a_5\frac{1}{T^5} + \dots \quad . \tag{33}$$

The coefficients are given by $a_1 = \frac{1}{48}$, $a_3 = \frac{7}{5760}$, $a_5 = \frac{31}{80640}$ and so on. A general formula for a_{2p-1}, $p\in\mathbb{Z}_{\geq 1}$, can either be derived via the methods given in [10, sections 6.5 and 6.7] or else via generalised Césaro methods - as we saw in [VIII] in the calculation of a_1 and in the calculation of asymptotic constants arising in the integrals for $\delta_1(T)$ and $\delta_2(T)$.

Since we need not just the general formula for a_{2p-1}, but also formulae for all of these integration constants as they arise successively in each $\delta_j(T)$, $j\in\mathbb{Z}_{\geq 1}$, we will adopt this latter generalised Césaro approach throughout the calculations which follow in this subsection.

From equation 33 we immediately have the following asymptotic expansions for the successive integrals of $\delta(T)$:

$$\begin{aligned}
\delta_1(T) &= \left\{a_1\ln(T) - \frac{a_3}{2}\frac{1}{T^2} - \frac{a_5}{4}\frac{1}{T^4} - \ldots\right\} + C_1^{(\delta)} \\
\delta_2(T) &= \left\{a_1(T\ln(T) - T) + \frac{a_3}{2!}\frac{1}{T} + \frac{a_5}{4\cdot 3}\frac{1}{T^3} + \ldots\right\} + \left\{C_1^{(\delta)}T + C_2^{(\delta)}\right\} \\
\delta_3(T) &= \left\{\begin{array}{c} \left\{a_1(\tfrac{1}{2}T^2\ln(T) - \tfrac{3}{4}T^2) + \tfrac{a_3}{2!}\ln(T) - \tfrac{a_5}{4\cdot 3\cdot 2}\tfrac{1}{T^2} - \ldots\right\} \\ + \left\{\tfrac{1}{2!}C_1^{(\delta)}T^2 + C_2^{(\delta)}T + C_3^{(\delta)}\right\} \end{array}\right\} \\
\delta_4(T) &= \left\{\begin{array}{c} \left\{a_1(\tfrac{1}{3!}T^3\ln(T) - \tfrac{11}{36}T^3) + \tfrac{a_3}{2!}(T\ln(T) - T) + \tfrac{a_5}{4!}\tfrac{1}{T} + \ldots\right\} \\ + \left\{\tfrac{1}{3!}C_1^{(\delta)}T^3 + \tfrac{1}{2!}C_2^{(\delta)}T^2 + C_3^{(\delta)}T + C_4^{(\delta)}\right\} \end{array}\right\} \\
&\vdots
\end{aligned} \tag{34}$$

and so on.

Now in equation 17 we have, by integration by parts, that

$$\frac{1}{\pi}\cdot\int_0^T t^n \mathrm{d}\delta(t) = \frac{1}{\pi}\cdot\left\{\begin{array}{c} T^n\delta(T) - nT^{n-1}\delta_1(T) + n(n-1)T^{n-2}\delta_2(T) \\ - \ldots + (-1)^n n!\delta_n(T) \end{array}\right\} \tag{35}$$

and similarly for $\frac{1}{\pi}\cdot\int_0^{\tilde{T}} \tilde{t}^n \mathrm{d}\delta(\tilde{t})$. It follows in equation 17, on combining with the formulae in equations 34, that we have

$$\overset{\circ}{C}_0 = \frac{1}{\pi}\cdot \underset{z,\tilde{z}\to\infty}{Clim}\left\{\{\delta(T) - \delta(0)\} + \left\{\delta(\tilde{T}) - \delta(0)\right\}\right\} = -\frac{1}{4} \tag{36}$$

and

$$\begin{aligned}
\overset{\circ}{C}_{-1} &= \frac{1}{\pi}\cdot \underset{z,\tilde{z}\to\infty}{Clim}\left\{\{T\delta(T) - \delta_1(T)\} - \left\{\tilde{T}\delta(\tilde{T}) - \delta_1(\tilde{T})\right\}\right\} \\
&= \frac{1}{\pi}\cdot \underset{z,\tilde{z}\to\infty}{Clim}\left\{\left\{a_1 - a_1\ln T - C_1^{(\delta)}\right\} - \left\{a_1 - a_1\ln\tilde{T} - C_1^{(\delta)}\right\}\right\} \\
&= \frac{1}{\pi}\cdot \underset{z,\tilde{z}\to\infty}{Clim}\left\{-a_1\ln T + a_1\ln\tilde{T}\right\}
\end{aligned} \tag{37}$$

and

$$\overset{\circ}{C}_{-2} = \frac{1}{\pi}\cdot \underset{z,\tilde{z}\to\infty}{Clim}\left\{\left\{\begin{array}{c} T^2\delta(T)-2T\delta_1(T) \\ \\ +\delta_2(T)\end{array}\right\}+\left\{\begin{array}{c} \tilde{T}^2\delta(\tilde{T})-2\tilde{T}\delta_1(\tilde{T}) \\ \\ +\delta_2(\tilde{T})\end{array}\right\}\right\}$$

$$= \frac{1}{\pi}\cdot \underset{z,\tilde{z}\to\infty}{Clim}\left\{\left\{-a_1T+2C_2^{(\delta)}\right\}+\left\{-a_1\tilde{T}+2C_2^{(\delta)}\right\}\right\}$$

$$= \frac{1}{\pi}\cdot \underset{z,\tilde{z}\to\infty}{Clim}\left\{-a_1T-a_1\tilde{T}+4C_2^{(\delta)}\right\} \qquad (38)$$

and in similar fashion

$$\overset{\circ}{C}_{-3} = \frac{1}{\pi}\cdot \underset{z,\tilde{z}\to\infty}{Clim}\left\{\left\{\begin{array}{c} -\frac{1}{2}a_1T^2-3a_3\ln T \\ \\ +\frac{11}{2}a_3-6C_3^{(\delta)}\end{array}\right\}-\left\{\begin{array}{c} -\frac{1}{2}a_1\tilde{T}^2-3a_3\ln\tilde{T} \\ \\ +\frac{11}{2}a_3-6C_3^{(\delta)}\end{array}\right\}\right\}$$

$$= \frac{1}{\pi}\cdot \underset{z,\tilde{z}\to\infty}{Clim}\left\{-\frac{1}{2}a_1T^2-3a_3\ln T+\frac{1}{2}a_1\tilde{T}^2+3a_3\ln\tilde{T}\right\} \qquad (39)$$

and

$$\overset{\circ}{C}_{-4} = \frac{1}{\pi}\cdot \underset{z,\tilde{z}\to\infty}{Clim}\left\{\left\{\begin{array}{c} -\frac{1}{3}a_1T^3-3a_3T \\ \\ +24C_4^{(\delta)}\end{array}\right\}+\left\{\begin{array}{c} -\frac{1}{3}a_1\tilde{T}^3-3a_3\tilde{T} \\ \\ +24C_4^{(\delta)}\end{array}\right\}\right\}$$

$$= \frac{1}{\pi}\cdot \underset{z,\tilde{z}\to\infty}{Clim}\left\{-\frac{1}{3}a_1T^3-3a_3T-\frac{1}{3}a_1\tilde{T}^3-3a_3\tilde{T}+48C_4^{(\delta)}\right\} \qquad (40)$$

and so forth.

Now since $z=(s_0-\frac{1}{2})-iT$ and $\tilde{z}=(s_0-\frac{1}{2})+i\tilde{T}$, we have $T=i(z-(s_0-\frac{1}{2}))$ and $\tilde{T}=-i(\tilde{z}-(s_0-\frac{1}{2}))$. From these, since powers z^ρ and $\tilde{z}^\rho$ are non-trivial eigenfunctions of P for $\rho\neq 0$, it follows at once within the geometric generalised Césaro framework that we have

$$T^n \overset{C}{\to} (-i)^n(s_0-\frac{1}{2})^n \quad \text{for all} \quad n\in\mathbb{Z}_{\geq 0} \qquad (41)$$

and

$$\tilde{T}^n \overset{C}{\to} (i)^n(s_0-\frac{1}{2})^n \quad \text{for all} \quad n\in\mathbb{Z}_{\geq 0}\,. \qquad (42)$$

In similar fashion, recalling the stipulation from [VIII] and [IX] that $\underset{z,\tilde{z}\to\infty}{Clim}\ln\left(\frac{z}{\tilde{z}}\right)=0$, and noting that $\ln T=\ln z+i\frac{\pi}{2}+o(1)$ and $\ln\tilde{T}=\ln\tilde{z}-i\frac{\pi}{2}+o(1)$, it also follows that

$$\ln\left(\frac{T}{\tilde{T}}\right) \overset{C}{\to} i\pi. \qquad (43)$$

Applying these results 41-43 in equations 37-40 and their extensions to $\overset{\circ}{C}_{-n}$ for higher n, we see that

$$\overset{\circ}{C}_{-1}=-i\cdot a_1 \quad \text{and} \quad \overset{\circ}{C}_{-3}=-i\cdot 3a_3 \quad \text{and} \quad \ldots$$

and

$$\overset{\circ}{C}_{-2} = \frac{1}{\pi} \cdot 2 \cdot 2! \cdot C_2^{(\delta)} \quad \text{and} \quad \overset{\circ}{C}_{-4} = \frac{1}{\pi} \cdot 2 \cdot 4! \cdot C_4^{(\delta)} \quad \text{and} \quad \ldots$$

and in general

$$\overset{\circ}{C}_{-n} = \begin{cases} -i \cdot n \cdot a_n & , \; n = 2p-1 \text{ odd} \\ \\ \frac{1}{\pi} \cdot 2 \cdot n! \cdot C_n^{(\delta)} & , \; n = 2p \text{ even}\,. \end{cases} \tag{44}$$

Here, in deriving the general result, we have observed that for n odd, the exclusively even pure powers which arise in T and $\tilde{T}$ have the same generalised Césaro limit and thus end up cancelling each other; while the same cancellation of Césaro limits of pure powers in T and $\tilde{T}$ occurs in the case of n even because in this case these powers are exclusively odd, with offsetting limiting values. When $n = 2k-1$ is odd this leaves only the limiting contribution from the residual terms in $\ln T$ and $\ln \tilde{T}$ (which arise solely from the terms $(2p-1)!\delta_{2p-1}(T)$ and $(2p-1)!\delta_{2p-1}(\tilde{T})$ in the computation), and we apply result 43 to these. When $n = 2p$ is even it leaves nothing but the re-inforcing terms in $C_{2p}^{(\delta)}$.

Using equation 44, we proceed by considering the cases of n odd and n even separately.

3.2.1 The case of $n = 2p-1$ odd

When $n = 2p-1$ ($p \in \mathbb{Z}_{>0}$) is odd, we need to calculate a general formula for a_{2p-1} and this only requires us to consider $\delta(T)$ itself, not any of its integrals, $\delta_j(T)$, $j \in \mathbb{Z}_{>0}$. We write $\delta(T)$ in equation 11 as

$$\delta(T) = H(T) + J(T) \tag{45}$$

where

$$H(T) = \frac{T}{4} \ln\left(1 + \frac{1}{4T^2}\right) + \frac{1}{4} \tan^{-1}\left(\frac{1}{2T}\right) \tag{46}$$

and

$$J(T) = \frac{T}{2} \int_0^\infty \frac{\overset{\vee}{q}_0(u)}{(u+\frac{1}{4})^2 + (\frac{T}{2})^2} \mathrm{d}u\,. \tag{47}$$

Contribution from $H(T)$: Now, in equation 46, we have directly from the well-known Taylor-series for $\ln(1+\epsilon)$ and $\tan^{-1}(\epsilon)$ that as $T \to \infty$

$$\begin{aligned} H(T) &= \frac{T}{4} \cdot \left\{ \frac{1}{(2T)^2} - \frac{1}{2}\frac{1}{(2T)^4} + \frac{1}{3}\frac{1}{(2T)^6} - \ldots \right\} \\ &\quad + \frac{1}{4} \cdot \left\{ \frac{1}{(2T)} - \frac{1}{3}\frac{1}{(2T)^3} + \frac{1}{5}\frac{1}{(2T)^5} - \ldots \right\} \\ &= \frac{1}{4} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{2^{2m-1}} \left\{ \frac{1}{2m} + \frac{1}{2m-1} \right\} \cdot \frac{1}{T^{2m-1}} \end{aligned} \tag{48}$$

so that the contribution to a_{2p-1} from $H(T)$ is given by

$$a_{2p-1}^{(H)} = \frac{(-1)^{p+1}}{2^{2p+1}} \left\{ \frac{1}{2p} + \frac{1}{2p-1} \right\} . \tag{49}$$

Contribution from $J(T)$: As for the contribution $a_{2p-1}^{(J)}$ from $J(T)$, although the integral defining $J(T)$ is classically convergent, we embrace generalised Césaro methods here to calculate its asymptotic expansion (as we did in [VIII] in the course of calculating $a_1^{(J)}$). This turns out to be much quicker (and we believe more elegant) than restricting to purely classical methods!

Specifically, we start with the formal expansion

$$\frac{1}{(u+\frac{1}{4})^2 + (\frac{T}{2})^2} = \frac{4}{T^2} \cdot \left\{ 1 - \frac{(2(u+\frac{1}{4}))^2}{T^2} + \frac{(2(u+\frac{1}{4}))^4}{T^4} - \dots \right\} \tag{50}$$

so that $a_{2p-1}^{(J)}$ is given by the classically divergent integral

$$a_{2p-1}^{(J)} = (-1)^{p-1} \cdot 2^{2p-1} \cdot \int_0^\infty \overset{\vee}{q}_0(u)\,(u+\frac{1}{4})^{2p-2}\,\mathrm{d}u \tag{51}$$

and we interpret its value as given by its generalised Césaro value, i.e.

$$a_{2p-1}^{(J)} = (-1)^{p-1} \cdot 2^{2p-1} \cdot \underset{X\to\infty}{\mathcal{C}lim} \int_0^X \overset{\vee}{q}_0(u)\,(u+\frac{1}{4})^{2p-2}\,\mathrm{d}u\,. \tag{52}$$

Now, to perform this latter calculation, we write $X = k + \alpha$ in the usual way and recall from [I]-[III] the fact that

$$\underset{X\to\infty}{\mathcal{C}lim} \sum_{j=1}^{k} j^\rho = \zeta(-\rho) \tag{53}$$

and also the key generalised Césaro asymptotic relation that for any $m, r \in \mathbb{Z}_{\geq 0}$, we have

$$\underset{X\to\infty}{\mathcal{C}lim}\, X^m \alpha^r \; = \begin{cases} \frac{1}{r+1} & , \quad m = 0 \\ \\ 0 & , \quad m > 0 \end{cases} \tag{54}$$

We also note that $\frac{\mathrm{d}}{\mathrm{d}u} \overset{\vee}{q}_0(u) = 1 - \sum_{j=1}^\infty \delta_j(u)$;[4] and that $\overset{\vee}{q}_0(X) = \alpha - \frac{1}{2}$ and $\overset{\vee}{q}_0(0) = -\frac{1}{2}$.

[4]Here, just this once, $\delta_j(u)$ refers of course to the Dirac delta-function centred at $u = j$ rather than having anything to do with our j^{th} iterated integral function arising from $\delta(T)$! We entreat the reader most earnestly to forgive this one-off notational confusion as merely a temporary awkwardness, rather than a more serious breach of manners.

It follows immediately by integration by parts and invocation of equations 53 and 54 that in general we have

$$\begin{aligned}
\int_0^X \overset{\vee}{q}_0(u)\,(u+\frac{1}{4})^m\,\mathrm{d}u &= \frac{1}{m+1}\left\{\begin{array}{l} \left[\overset{\vee}{q}_0(u)\,(u+\frac{1}{4})^{m+1}\right]_0^X \\ -\int_0^X (u+\frac{1}{4})^{m+1}\,\mathrm{d}u \\ +\sum_{j=1}^k (j+\frac{1}{4})^{m+1} \end{array}\right\} \\
&= \frac{1}{m+1}\left\{\begin{array}{l} \left[(X+\frac{1}{4})^{m+1}(\alpha-\frac{1}{2})+\frac{1}{2^{2m+3}}\right] \\ -\frac{1}{m+2}\left[X+\frac{1}{4})^{m+2}-(\frac{1}{4})^{m+2}\right] \\ +\sum_{l=0}^{m+1}\binom{m+1}{l}(\frac{1}{4})^l\sum_{j=1}^k j^{m-l+1} \end{array}\right\} \\
&\overset{C}{\to} \frac{1}{(m+1)}\frac{1}{2^{2m+3}} \\
&\quad + \frac{1}{(m+1)}\sum_{l=0}^{m+1}\binom{m+1}{l}\frac{1}{2^{2l}}\,\underset{X\to\infty}{Clim}\sum_{j=1}^{k} j^{m-l+1} \\
&= \frac{1}{(m+1)}\frac{1}{2^{2m+3}} \\
&\quad + \frac{1}{(m+1)}\sum_{l=0}^{m+1}\binom{m+1}{l}\frac{1}{2^{2l}}\zeta(l-m-1)\,. \qquad (55)
\end{aligned}$$

Now for $l-m-1<0$, $\zeta(l-m-1)$ is related to the Bernoulli numbers[5] by $\zeta(l-m-1)=-\frac{1}{m-l+2}B_{m-l+2}$. Hence after cancellation of the first term with the $l=m+1$ term in the second sum, equation 54 may be re-expressed as the generalised Césaro evaluation

$$\int_0^\infty \overset{\vee}{q}_0(u)\,(u+\frac{1}{4})^m\,\mathrm{d}u = -\frac{1}{(m+1)}\cdot\sum_{l=0}^{m}\frac{\binom{m+1}{l}}{m-l+2}\,B_{m-l+2}\,\frac{1}{2^{2l}}\,. \qquad (56)$$

In equation 52 it follows that we have

$$a_{2p-1}^{(J)} = (-1)^p\cdot 2^{2p-1}\cdot\frac{1}{(2p-1)}\cdot\sum_{l=0}^{2p-2}\frac{\binom{2p-1}{l}}{2p-l}\,B_{2p-l}\,\frac{1}{2^{2l}}\quad . \qquad (57)$$

Combining the contributions: Hence overall, combining the contributions

[5]defined as in [I]-[III] in alternating fashion, so that $B_1=-\frac{1}{2}$, $B_2=\frac{1}{6}$, $B_3=0$, $B_4=-\frac{1}{30}$ and so on

from $a^{(H)}_{2p-1}$ and $a^{(J)}_{2p-1}$, we have that

$$\begin{aligned} a_{2p-1} &= \frac{(-1)^{p+1}}{2^{2p+1}} \left\{ \frac{1}{2p} + \frac{1}{2p-1} \right\} \\ & \quad +(-1)^p \cdot \frac{2^{2p-1}}{(2p-1)} \cdot \sum_{l=0}^{2p-2} \frac{\binom{2p-1}{l}}{2p-l} B_{2p-l} \frac{1}{2^{2l}} \end{aligned} \tag{58}$$

(which we can readily check agrees with the values of a_1, a_3 and a_5 given earlier).

It finally follows in equation 44 that, for $n = 2p-1$ odd, we have

$$\overset{\circ}{C}_{-(2p-1)} = -i \cdot (2p-1) \cdot a_{2p-1} \tag{59}$$

where a_{2p-1} is as just given in equation 58.

3.2.2 The case of $n = 2p$ even

Turning now to the case of $n = 2p$ even ($p \in \mathbb{Z}_{>0}$), the first thing we see from equation 44 is that we need to consider not just $\delta(T)$, but all its successive integrals, in order to find the general form of the asymptotic constants of integration $C^{(\delta)}_{2p}$. However, we saw in [VIII] that even calculating $C^{(\delta)}_1$ and $C^{(\delta)}_2$ is already a lengthy and elaborate business when performed in full, and continuing in the same fashion using the full detailed form of the anti-derivative at each step rapidly becomes unworkable.

To proceed to higher $C^{(\delta)}_n$, at least for n even, the key insight which renders calculation tractable is the following. From the form of the expressions for $\delta_n(T)$ in equation 34 we see that we can isolate $C^{(\delta)}_n$ by

(i) Ignoring (effectively setting to 0) all terms of the form T^m or $T^m \ln T$ for any $m \in \mathbb{Z}_{\geq 1}$, and then

(ii) Taking the limit as $T \to \infty$ in the residual expression.

In the latter of these steps, the limit required is in principle just a classical limit. However, when we engage in Césaro computations in order to simplify calculations, as we did in the previous subsection - particularly with respect to $J(T)$ (see e.g. section 3.2.4) - this will naturally become a (strong) generalised Césaro limit.

As regards the first step, let $\mathcal{P}$ be the space of functions spanned by finite linear combinations from the sets of functions $\{t^m \ln t\}_{m=1}^{\infty}$ and $\{t^m\}_{m=1}^{\infty}$, i.e. $\mathcal{P} = \left\{ f \mid f(t) = \sum_{i=1}^{M} a_i t^{\mu_i} \ln t + \sum_{j=1}^{N} b_j t^{\nu_j} \right\}$ for some collection of real constants, a_i and b_j , and some collection of positive integers, μ_i and ν_j; and let $\mathbf{I}$ be the integration operator, $\mathbf{I}[g](T) := \int_0^T g(t)\, \mathrm{d}t$.

Then it is obvious that $\mathcal{P}$ is closed under the action of $\mathbf{I}$ and it follows at once that not only is step (i) equivalent to considering $\delta_n(T)$ modulo $\mathcal{P}$ but that, since $\delta_n(T) = \mathbf{I}^n[\delta](T)$, so it is permissible to work modulo $\mathcal{P}$ throughout the process of repeatedly applying $\mathbf{I}$ in order to calculate $\delta_n(T)$ from $\delta(T)$.

This then radically simplifies our working. Rather than becoming unmanageably messy very quickly, if we work modulo $\mathcal{P}$ we can instead derive simple integral recurrence relations which allow us to readily perform the iterated integrations and calculate the required formula for $C_n^{(\delta)}$ from $\delta_n(T)$, at least when n is even.

Specifically, we will again consider separately the contributions to $C_{2k}^{(\delta)}$ from $H(T)$ and from $J(T)$, denoting these by $C_{2k}^{(\delta,H)}$ and $C_{2k}^{(\delta,J)}$ respectively; and we will start with the case of $H(T)$.

3.2.3 An expression for $C_{2k}^{(\delta,H)}$, $k \in \mathbb{Z}_{\geq 1}$

In this case, we can rewrite $H(T)$ modulo $\mathcal{P}$ as

$$H(T) = \tilde{H}(T) + \frac{\pi}{8} \quad \text{where} \quad \tilde{H}(T) = \frac{T}{4}\ln(1+4T^2) - \frac{1}{4}\tan^{-1}(2T) \tag{60}$$

and since integration immediately also sends the constant term, $\frac{\pi}{8}$, into $\mathcal{P}$ so we need only consider $\tilde{H}(T)$ in calculating our expression for $\mathbf{I}^{2k}[H](T)$ modulo $\mathcal{P}$.

Now, letting $\tilde{T}_m$ and $\tilde{L}_m$ ($m \in \mathbb{Z}_{\geq 0}$) be defined by

$$\tilde{T}_m = \tilde{T}_m(T) := T^m \tan^{-1}(2T) \quad \text{and} \quad \tilde{L}_m = \tilde{L}_m(T) := T^m \ln(1+4T^2) \tag{61}$$

we have the following lemmas:

Lemma 1: *$\tilde{T}_m$ and $\tilde{L}_m$ satisfy the following recurrence relations modulo $\mathcal{P}$:*

$$\begin{aligned}\mathbf{I}^2[\tilde{T}_{2k}] &= \frac{1}{(2k+1)(2k+2)}\tilde{T}_{2k+2} + \frac{(-1)^{k+1}}{(2k+1)}\frac{1}{2^{2k+2}}\tilde{L}_1 \\ &\quad + \frac{(-1)^{k+1}}{(2k+2)}\frac{1}{2^{2k+2}}\tilde{T}_0 \qquad (mod\mathcal{P})\end{aligned} \tag{62}$$

and

$$\begin{aligned}\mathbf{I}^2[\tilde{L}_{2k-1}] &= \frac{1}{(2k)(2k+1)}\tilde{L}_{2k+1} + \frac{(-1)^{k+1}}{(2k)}\frac{1}{2^{2k}}\tilde{L}_1 \\ &\quad + \frac{(-1)^{k+1}}{(2k+1)}\frac{1}{2^{2k}}\tilde{T}_0 \qquad (mod\mathcal{P})\ .\end{aligned} \tag{63}$$

Lemma 2: *For all $k \in \mathbb{Z}_{\geq 1}$ we have*

$$\tilde{L}_{2k-1} = o(1) \qquad (mod\mathcal{P}) \tag{64}$$

and

$$\tilde{T}_{2k} = o(1) \qquad (mod\mathcal{P}) \tag{65}$$

and for $k=0$ *we have*

$$\tilde{T}_0 = \frac{\pi}{2} + o(1) \qquad (mod\mathcal{P}) \ . \tag{66}$$

Proofs: The key which makes lemma 1 work is the fact that the asymptotic expansion for $\frac{1}{1+4T^2}$ is "self-replicating" in the sense that, for any $j \in \mathbb{Z}_{\geq 0}$ we can truncate this asymptotic expansion at the j^{th} term and collapse the tail into a single term involving $\frac{1}{1+4T^2}$ again, i.e. we have

$$\frac{1}{1+4T^2} = \frac{1}{4T^2} \cdot \left\{ \left\{ 1 - \frac{1}{4T^2} + \ldots + \frac{(-1)^j}{2^{2j}T^{2j}} \right\} + \frac{(-1)^{j+1}}{2^{2j}T^{2j}} \frac{1}{1+4T^2} \right\} . \tag{67}$$

It follows that we have, by integration by parts,

$$\begin{aligned} \mathbf{I}[\tilde{T}_{2k}] \quad = \quad & \frac{1}{(2k+1)} \tilde{T}_{2k+1} \\ & - \frac{1}{2(2k+1)} \int_0^T t^{2k-1} \cdot \left\{ 1 - \frac{1}{4t^2} + \ldots + \frac{(-1)^{k-1}}{2^{2k-2}t^{2k-2}} \right\} \mathrm{d}t \\ & + \frac{(-1)^{k+1}}{2(2k+1)2^{2k-2}} \int_0^T \frac{t}{1+4t^2} \, \mathrm{d}t \end{aligned}$$

and since the collection of terms in the first integral all integrate to functions in $\mathcal{P}$, so

$$\mathbf{I}[\tilde{T}_{2k}] = \frac{1}{(2k+1)} \tilde{T}_{2k+1} + \frac{(-1)^{k+1}}{(2k+1)2^{2k+2}} \tilde{L}_0 \qquad (mod\mathcal{P}) \ .$$

Applying $\mathbf{I}$ again, once again invoking equation 67 (this time with $j=k$ rather than $j=k-1$) and working in the same fashion modulo $\mathcal{P}$, we are then able to readily deduce equation 62 after elementary simplifications.

The proof of equation 63 follows in identical fashion and this completes the proof of lemma 1.

Lemma 2 in turn follows immediately from the fact that, on the one hand, $\ln(1+4T^2) = 2\ln T + 2\ln 2 + \frac{1}{4T^2} - \frac{1}{32T^4} + \ldots$ with only *even* powers of $\frac{1}{T}$ in the asymptotic series as $T \to \infty$; while on the other, $\tan^{-1}(T) = \frac{\pi}{2} - \tan^{-1}\left(\frac{1}{T}\right) = \frac{\pi}{2} - \frac{1}{2T} + \frac{1}{3}\frac{1}{8T^3} - \ldots$ has only odd powers in $\frac{1}{T}$ as $T \to \infty$; while finally, $\tilde{T}_0(T) = \tan^{-1}(2T) = \frac{\pi}{2} + o(1)$.

With these two lemmas it is now straightforward to perform step (ii) and calculate $C_{2k}^{(\delta,H)}$.

Since $\tilde{H}(T) = \frac{1}{4}\tilde{L}_1 - \frac{1}{4}\tilde{T}_0$, it follows at once by lemma 1 that modulo $\mathcal{P}$ we have

$$\mathbf{I}^2[\tilde{H}] = \frac{1}{24}\tilde{L}_3 + \frac{3}{32}\tilde{L}_1 - \frac{1}{8}\tilde{T}_2 + \frac{5}{96}\tilde{T}_0 \tag{68}$$

and thus, by lemma 2, $C_2^{(\delta,H)} = \frac{5\pi}{192} = \frac{(4\cdot 1+1)}{2^5\cdot 3!}\,\pi$.

In the same fashion, applying $\mathbf{I}^2$ in equation 68 and using lemma 1, we have modulo $\mathcal{P}$ that

$$\mathbf{I}^4[\tilde{H}] = \frac{1}{480}\,\tilde{L}_5 + \frac{1}{64}\,\tilde{L}_3 - \frac{7}{3\cdot 2^9}\,\tilde{L}_1 - \frac{1}{96}\,\tilde{T}_4 + \frac{5}{192}\,\tilde{T}_2 - \frac{3}{5\cdot 2^9}\,\tilde{T}_0$$

so that, by lemma 2, $C_4^{(\delta,H)} = -\frac{3\pi}{5120} = -\frac{(4\cdot 2+1)}{2^7\cdot 5!}\,\pi$. Continuing in the same way we calculate that $C_6^{(\delta,H)} = \frac{(4\cdot 3+1)}{2^9\cdot 7!}\,\pi$ and in general

$$C_{2k}^{(\delta,H)} = \frac{(-1)^{k+1}(4\cdot k+1)}{2^{2k+3}\cdot(2k+1)!}\,\pi\,. \tag{69}$$

3.2.4 An expression for $C_{2k}^{(\delta,J)}$, $k\in\mathbb{Z}_{\geq 1}$

Turning now to the contribution from $J(T)$, $C_{2k}^{(\delta,J)}$, in this case the natural quantities to introduce are

$$\tilde{L}_{m,n} = \tilde{L}_{m,n}(T) := \int_0^\infty \overset{\vee}{q}_0(u)\,(u+\frac{1}{4})^m\,T^n\,\ln\left((u+\frac{1}{4})^2+\frac{T^2}{4}\right)\,\mathrm{d}u \tag{70}$$

and

$$\tilde{T}_{m,n} = \tilde{T}_{m,n}(T) := \int_0^\infty \overset{\vee}{q}_0(u)\,(u+\frac{1}{4})^m\,T^n\,\tan^{-1}\left(\frac{T}{2(u+\frac{1}{4})}\right)\,\mathrm{d}u \tag{71}$$

and the analogues of lemmas 1 and 2 are as follows:

Lemma $1'$**:** *$\tilde{T}_{m,n}$ and $\tilde{L}_{m,n}$ satisfy the following recurrence relations modulo $\mathcal{P}$:*

$$\begin{aligned}\mathbf{I}^2[\tilde{T}_{m,2k}] \quad = \quad & \frac{1}{(2k+1)(2k+2)}\,\tilde{T}_{m,2k+2} + \frac{(-1)^{k+1}}{(2k+1)}\,2^{2k}\,\tilde{L}_{m+2k+1,1} \\ & + \frac{(-1)^{k+1}}{(2k+2)}\,2^{2k+2}\,\tilde{T}_{m+2k+2,0} \qquad (mod\mathcal{P})\end{aligned} \tag{72}$$

and

$$\begin{aligned}\mathbf{I}^2[\tilde{L}_{m,2k-1}] \quad = \quad & \frac{1}{(2k)(2k+1)}\,\tilde{L}_{m,2k+1} + \frac{(-1)^{k+1}}{(2k)}\,2^{2k}\,\tilde{L}_{m+2k,1} \\ & + \frac{(-1)^{k+1}}{(2k+1)}\,2^{2k+2}\,\tilde{T}_{m+2k+1,0} \qquad (mod\mathcal{P})\,.\end{aligned} \tag{73}$$

Lemma $2'$**:** *For any $k\in\mathbb{Z}_{\geq 1}$ and $j\in\mathbb{Z}_{\geq 0}$ with $j<k$ we have*

$$\tilde{L}_{2j,2k-2j-1} = o(1) \qquad (mod\mathcal{P}) \tag{74}$$

and

$$\tilde{T}_{2j+1,2k-2j-2} = o(1) \qquad (mod\mathcal{P}) \qquad \text{provided} \qquad 2k-2j-2 \neq 0\,. \qquad (75)$$

In the latter result, if $2k-2j-2=0$ *then we have the strong Césaro limit*

$$\tilde{T}_{2k-1,0} \overset{C}{\simeq} -\frac{\pi}{2}\cdot\frac{1}{2k}\sum_{l=0}^{2k-1}\frac{\binom{2k}{l}}{2k+1-l}\,B_{2k+1-l}\,\frac{1}{2^{2l}}\,. \qquad (76)$$

Proofs: The proof of lemma $1'$ follows in exactly the same fashion as for lemma 1. The only material amendments are to adapt the self-replication identity 67 to say that, for any $j \in \mathbb{Z}_{\geq 0}$

$$\frac{1}{(u+\frac{1}{4})^2+\frac{T^2}{4}} = \frac{4}{T^2}\cdot\left\{\begin{array}{c} \left\{1-\frac{4(u+\frac{1}{4})^2}{T^2}+\ldots+\frac{(-1)^j 2^{2j}(u+\frac{1}{4})^{2j}}{T^{2j}}\right\} \\ \\ +\frac{(-1)^{j+1}2^{2j}(u+\frac{1}{4})^{2j+2}}{T^{2j}}\;\frac{1}{(u+\frac{1}{4})^2+\frac{T^2}{4}} \end{array}\right\} \qquad (77)$$

and to note that we may reverse the order of integration every time we apply the operator $\mathbf{I}$ throughout our working, while leaving coefficients (of terms in T) which are themselves strongly Césaro-convergent integrals in u.

As regards lemma $2'$, it also follows in identical fashion to lemma 2 from consideration of the evenness or oddness of the decaying terms (in T) in the asymptotic series for $\ln\left((u+\frac{1}{4})^2+\frac{T^2}{4}\right)$ and $\tan^{-1}\left(\frac{T}{2(u+\frac{1}{4})}\right)$; and from the resulting implications when working modulo $\mathcal{P}$ (as a space of functions in T).

The final identity 76 follows at once from the fact that $\tilde{T}_{2k-1,0} \overset{C}{\simeq} \frac{\pi}{2}\cdot \int_0^\infty \overset{\vee}{q}_0(u)\,(u+\frac{1}{4})^{2k-1}\,\mathrm{d}u$ (on taking the strong Césaro limit inside the integral) and recalling equation 56.

Since, by direct computation, we have, modulo $\mathcal{P}$, that

$$\mathbf{I}^2[J] = \tilde{L}_{0,1} + 4\,\tilde{T}_{1,0}$$

it follows by lemma $1'$ that we also have, modulo $\mathcal{P}$, that

$$\mathbf{I}^4[J] = \frac{1}{3!}\,\tilde{L}_{0,3} - 2\,\tilde{L}_{2,1} + 2\,\tilde{T}_{1,2} - \frac{8}{3}\,\tilde{T}_{3,0}$$

and

$$\mathbf{I}^6[J] = \frac{1}{5!}\,\tilde{L}_{0,5} - \frac{1}{3}\,\tilde{L}_{2,3} + \frac{2}{3}\,\tilde{L}_{4,1} + \frac{1}{6}\,\tilde{T}_{1,4} - \frac{4}{3}\,\tilde{T}_{3,2} + \frac{8}{15}\,\tilde{T}_{5,0}$$

and so on, with the coefficient of $\tilde{T}_{2k-1,0}$ in $\mathbf{I}^{2k}[J]$ being $(-1)^{k+1}\frac{2^{2k}}{(2k-1)!}$. It thus follows immediately, by lemma $2'$, that in general

$$C_{2k}^{(\delta,J)} = \pi\cdot(-1)^k\frac{2^{2k-1}}{(2k)!}\cdot\sum_{l=0}^{2k-1}\frac{\binom{2k}{l}}{2k+1-l}\,B_{2k+1-l}\,\frac{1}{2^{2l}}\,. \qquad (78)$$

3.2.5 Final expressions for $C_{2p}^{(\delta)}$ and $\overset{\circ}{C}_{-2p}$

Combining our formulae for $C_{2p}^{(\delta,H)}$ and $C_{2p}^{(\delta,J)}$ in equations 69 and 78, we then obtain at once a final expression for $C_{2p}^{(\delta)}$, namely

$$C_{2p}^{(\delta)} = \pi \cdot \left\{ \begin{array}{c} \frac{(-1)^{p+1}\cdot(4\cdot p+1)}{2^{2p+3}\cdot(2p+1)!} \\ + (-1)^p \, \frac{2^{2p-1}}{(2p)!} \cdot \sum_{l=0}^{2p-1} \frac{\binom{2p}{l}}{2p+1-l} \, B_{2p+1-l} \, \frac{1}{2^{2l}} \end{array} \right\} . \quad (79)$$

It then follows in equation 44 that for $n = 2p$ even, we also have a final expression for $\overset{\circ}{C}_{-2p}$, namely

$$\begin{aligned} \overset{\circ}{C}_{-2p} &= 2\cdot(2p)!\cdot(-1)^{p+1}\cdot \left\{ \begin{array}{c} \frac{(4\cdot p+1)}{2^{2p+3}\cdot(2p+1)!} \\ - \frac{2^{2p-1}}{(2p)!} \cdot \sum_{l=0}^{2p-1} \frac{\binom{2p}{l}}{2p+1-l} \, B_{2p+1-l} \, \left(\frac{1}{2^{2l}}\right) \end{array} \right\} \\ &= 2\cdot(-1)^{p+1}\cdot \left\{ \begin{array}{c} \frac{(4\cdot p+1)}{2^{2p+3}\cdot(2p+1)} \\ - 2^{2p-1} \cdot \sum_{l=0}^{2p-1} \frac{\binom{2p}{l}}{2p+1-l} \, B_{2p+1-l} \, \left(\frac{1}{2^{2l}}\right) \end{array} \right\} . \quad (80) \end{aligned}$$

Between equations 58 and 59, on the one hand, and equation 80 on the other, we thus now have final expressions for $\overset{\circ}{C}_{-n}$ for all $n \in \mathbb{Z}_{\geq 1}$, odd or even.

Comment: In subsections 3.2.1 - 3.2.5 we have seen the utility of, on the one hand, fully embracing strong Césaro computation (in the u-variable) and on the other, working modulo $\mathcal{P}$ (in the T-variable) when iteratively integrating $\delta(T)$.

The specific form of $\delta(T)$ was, of course, critical in enabling the success of this two-pronged approach to calculation of its asymptotic coefficients, a_{2p-1}, and integration constants, $C_{2p}^{(\delta)}$. In particular, the fact that the integral defining $J(T)$ involves $\overset{\vee}{q}_0(u)$, which is the base of a Césaro-adapted scale, is what facilitates the wholesale use of generalised Césaro computation; while the fact that throughout the iterated integration of $H(T)$ and $J(T)$ we repeatedly encounter expressions of the form $\frac{1}{a^2+b^2T^2}$ which yield "self-replicating" asymptotic expansions, is what enables the required derivation of simple recurrence relations when working modulo $\mathcal{P}$.

Despite this particularity, however, we nonetheless feel that one or both of these elements may be adaptable to other situations where the gentle reader may be seeking either the coefficients of an asymptotic expansion of some function, or the constants of integration of its iterated integrals.

3.3 A general formula for $\overset{\circ}{B}_{-n}$

Since we are assuming RH so, as discussed in [VIII], $S(T)$ forms the base of a Césaro-adapted scale of functions $\{S_n^*(T)\}_{n=0}^{\infty}$. It follows (see [VIII], result 4) that we have the strong Césaro relationship that

$$T^n S_r^*(T) \overset{C}{\simeq} 0 \qquad \text{as} \quad T \to \infty \tag{81}$$

via P^{n+1}, i.e.

$$P^{n+1}\left[t^n S_r^*(t)\right](T) = o(1)\,. \tag{82}$$

We also saw in [VIII] that we have the following explicit relationships between the $\{S_n^*(T)\}_{n=0}^{\infty}$ and the iterated integrals, $\{S_n(T)\}_{n=0}^{\infty}$ which we defined earlier:

$$\begin{aligned} S_1(T) &= S_1^*(T) + C_S^{(1)} \\ S_2(T) &= S_2^*(T) + C_S^{(1)} T + C_S^{(2)} \\ S_3(T) &= S_3^*(T) + \frac{1}{2!}\, C_S^{(1)}\, T^2 + C_S^{(2)} T + C_S^{(3)} \end{aligned}$$

and in general, if we set $C_S^{(0)} := 0$, then

$$S_n(T) = S_n^*(T) + \sum_{i=0}^{n} \frac{1}{(n-i)!}\, C_S^{(i)}\, T^{n-i}\,. \tag{83}$$

Now, in equation 16 (in identical fashion as before in the case of $\delta(T)$) we deduce by integration by parts that

$$\int_0^T t^n \mathrm{d}S(t) = \left\{ \begin{array}{c} T^n S(T) - nT^{n-1} S_1(T) + n(n-1)T^{n-2} S_2(T) \\ - \ldots + (-1)^n n! S_n(T) \end{array} \right\}. \tag{84}$$

Applying the formula just given in equation 83, we can re-express this equation with the functions $S_j(T)$ replaced by the functions $S_j^*(T)$, together with a collection of pure powers of T.

But in this latter collection, the coefficient of T^n is immediately 0; and the coefficient of T^{n-1} is

$$\begin{aligned} coefft_{T^{n-1}} &= \left\{ \begin{array}{c} -n + n(n-1) - n(n-1)(n-2) \cdot \frac{1}{2!} + \ldots \\ +(-1)^j \frac{n!}{(n-j)!} \cdot \frac{1}{(j-1)!} + \ldots + (-1)^n \frac{n!}{0!} \cdot \frac{1}{(n-1)!} \end{array} \right\} \cdot C_S^{(1)} \\ &= C_S^{(1)} \cdot \sum_{j=0}^{n} (-1)^j \cdot j \cdot \binom{n}{j} \\ &= C_S^{(1)} \cdot \left(\frac{\mathrm{d}}{\mathrm{d}a} (1-a)^n \right) \Bigg|_{a=1} = 0\,. \end{aligned}$$

Likewise, for $2 \leq k \leq n-1$, we have that the coefficient of T^{n-k} is

$$coefft_{T^{n-k}} = C_S^{(k)} \cdot \left(\left(\frac{\mathrm{d}}{\mathrm{d}a} \right)^k (1-a)^n \right) \Bigg|_{a=1} = 0$$

while, finally, for $k = n$ the constant term does not cancel and is simply given by

$$coefft_{T^0} = (-1)^n \cdot n! \cdot C_S^{(n)} \quad .$$

Thus equation 84 becomes re-expressed as

$$\int_0^T t^n \mathrm{d}S(t) = \left\{ \begin{array}{c} T^n S_0^*(T) - nT^{n-1} S_1^*(T) + n(n-1) T^{n-2} S_2^*(T) \\ - \ldots + (-1)^n n! S_n^*(T) \end{array} \right\} + (-1)^n \cdot n! \cdot C_S^{(n)} \tag{85}$$

and in light of results 81 and 82, we see that we have the strong Césaro relationship that

$$\int_0^T t^n \mathrm{d}S(t) \overset{C}{\simeq} (-1)^n \cdot n! \cdot C_S^{(n)} \qquad \text{as} \quad T \to \infty \tag{86}$$

via P^{n+1}. This implies that we have the following strong Césaro integral:

$$\int_0^\infty t^n \mathrm{d}S(t) = (-1)^n \cdot n! \cdot C_S^{(n)} . \tag{87}$$

In equation 16 it follows at once, finally, that for any $n \in \mathbb{Z}_{\geq 1}$ we have the following expression for $\overset{\circ}{B}_{-n}$:

$$\begin{aligned} \overset{\circ}{B}_{-n} &= (-1)^n \cdot n! \cdot C_S^{(n)} \cdot \{1 + (-1)^n\} \\ &= \begin{cases} 2 \cdot (2p)! \cdot C_S^{(2p)} , & n = 2p \quad \text{even} \\ 0 , & n \quad \text{odd.} \end{cases} \end{aligned} \tag{88}$$

3.4 A final expression for $\sum_{\{s_0-NT\}} M_i \gamma_i^n$ and $r_{NT}(s_0, -n)$

Combining the formulae we have derived in this section for $\overset{\circ}{A}_{-n}$, $\overset{\circ}{B}_{-n}$ and $\overset{\circ}{C}_{-n}$, we are now in a position to give final formulae for $\sum_{\{s_0-NT\}} M_i \gamma_i^n$. We have that when $n = (2p-1)$ is odd

$$\sum_{\{s_0-NT\}} M_i \gamma_i^n = \left\{ \begin{array}{c} \frac{(-i)^{2p-1}}{4p}\,(s_0 - \frac{1}{2})^{2p} \\ \\ -i \cdot \left\{ \begin{array}{c} \frac{(-1)^{p+1}}{2^{2p+1}}\,\left\{2 - \frac{1}{2p}\right\} \\ \\ +(-1)^p \cdot 2^{2p-1} \cdot \sum_{l=0}^{2p-2} \frac{\binom{2p-1}{l}}{2p-l}\, B_{2p-1}\, \frac{1}{2^{2l}} \end{array} \right\} \end{array} \right\} \tag{89}$$

and when $n = 2p$ is even

$$\sum_{\{s_0-NT\}} M_i \gamma_i^n = \left\{ \begin{array}{c} \frac{(-1)^p}{2(2p+1)}\,(s_0 - \frac{1}{2})^{2p+1} \\ \\ -2 \cdot (-1)^p \left\{ \begin{array}{c} \frac{(4 \cdot p+1)}{2^{2p+3} \cdot (2p+1)} \\ \\ -\,2^{2p-1} \cdot \sum_{l=0}^{2p-1} \frac{\binom{2p}{l}}{2p+1-l}\, B_{2p+1-l}\, \left(\frac{1}{2^{2l}}\right) \end{array} \right\} \\ \\ +2 \cdot (2p)! \cdot C_S^{(2p)} \end{array} \right\} \tag{90}$$

and these then give us the final formula we are seeking for $r_{NT}(s_0, -n)$ when applied in equation 7.

Comments: (i) Note that in these expressions for $\sum_{\{s_0-NT\}} M_i \gamma_i^n$ the contribution from $\overset{\circ}{A}_{-n}$ has explicit s_0-dependence because it arises from a *geometric* Césaro limit, where the *location* of the non-trivial roots has been crucial and has necessitated the removal of non-trivial eigenfunctions in the geometric Césaro variables (z, $\tilde{z}$ or w, $\tilde{w}$) in the course of calculation. The geometric interactions, and in particular the balancing and cancellation of terms arising from roots above and below the real axis (which would otherwise lead to simple poles in s_0) has been delicate, however, and leaves a much simpler s_0-dependence than might be expected, namely a single simple power in $(s_0 - \frac{1}{2})^{n+1}$.

By contrast, the contributions from $\overset{\circ}{C}_{-n}$ and $\overset{\circ}{B}_{-n}$, while intricate, are both independent of s_0. This reflects the fact that both arise from *strong* Césaro calculations requiring only repeated averaging (i.e. application of pure powers of P alone, with no associated removal of geometric eigenfunctions of P) in their handling of the integrals involving $\delta(t)$ and $S(t)$.

(ii) As a consequence it follows, when combined with the form of equation 7, that for $n \in \mathbb{Z}_{\geq 0}$, $r_{NT}(s_0, -n)$ is always a *polynomial* in $(s_0 - \frac{1}{2})$.

4 Implications of the generalised root identities for ζ at $\mu = -n$: A new set of identities for $S(t)$ conditional on RH

We now return to applying equation 2, giving the generalised root identities for ζ at $\mu = -n$, $n \in \mathbb{Z}_{\geq 0}$, to see whether they imply any new results regarding ζ.

Now, since it is immediate from equations 3 and 5 that $r_P(s_0, -n)$ and $r_T(s_0, -n)$ are also both polynomials in s_0 for any $n \in \mathbb{Z}_{\geq 0}$, it follows from the last remark in the previous section that $r_\zeta(s_0, -n) = r_P(s_0, -n) + r_T(s_0, -n) + r_{NT}(s_0, -n)$ is likewise a polynomial in s_0 for all such n.

Thus $r_\zeta(s_0, -n)$ extends at once from a function on $Re(s_0) > 1$ (the s_0-domain to which we had restricted in [IX] in establishing $r_\zeta(s_0, \mu)$ as an analytic function in μ for $\mu \in \mathbb{C} \setminus \{1\}$) to all $s_0 \in \mathbb{C}$.

In light of this, let us first simplify things by applying equation 2 not at arbitrary s_0, but just at $s_0 = \frac{1}{2}$, i.e.

$$r_\zeta(\frac{1}{2}, -n) = r_P(\frac{1}{2}, -n) + r_T(\frac{1}{2}, -n) + r_{NT}(\frac{1}{2}, -n) = 0\,. \tag{91}$$

In this case our formula for $r_{NT}(s_0, -n)$ in equation 7 collapses down to just the constant term

$$r_{NT}(\frac{1}{2}, -n) = i^n \cdot \sum_{\{s_0 - NT\}} M_i \gamma_i^n \quad . \tag{92}$$

Here $\sum_{\{s_0 - NT\}} M_i \gamma_i^n$ is given by either equation 89 or equation 90 depending on whether n is odd or even, and in either case the leading term containing the power $(s_0 - \frac{1}{2})^{n+1}$ vanishes, leaving only the residual constant pieces.

The formulae for $r_P(\frac{1}{2}, -n)$ and $r_T(\frac{1}{2}, -n)$ likewise simplify as

$$r_P(\frac{1}{2}, -n) = -\frac{1}{2^n} \tag{93}$$

and

$$r_T(\frac{1}{2}, -n) = \frac{(-1)^{n+1}}{2^n} \cdot \left\{ \frac{1}{4(n+1)} + \frac{1}{2} - \sum_{j=1}^{n} \binom{n}{j} \zeta(-j)\, 2^{2j} \right\} . \tag{94}$$

We now consider the cases of n odd and n even in turn.

4.1 The case of $n = 2p - 1$ odd

In this case, from equation 89, we have in equation 91 that

$$LHS \quad = \quad \left\{ \begin{array}{c} \frac{-1}{2^{2p-1}} + \frac{1}{2^{2p-1}} \cdot \left\{ \frac{1}{8p} + \frac{1}{2} - \sum_{j=1}^{2p-1} \binom{2p-1}{j} \zeta(-j)\, 2^{2j} \right\} \\ \\ + \left\{ \begin{array}{c} \frac{1}{2^{2p+1}} \left\{ 2 - \frac{1}{2p} \right\} \\ \\ -2^{2p-1} \cdot \sum_{l=0}^{2p-2} \binom{2p-1}{l} \frac{B_{2p-l}}{(2p-l)} \frac{1}{2^{2l}} \end{array} \right\} \end{array} \right\} .$$

Now, on recalling that $\zeta(-j) = -\frac{B_{j+1}}{(j+1)}$ for all $j \in \mathbb{Z}_{\geq 0}$ and changing summation variable in the second sum from l to $j = 2p-1-l$, we find that the two sums in this expression cancel; while the remaining terms also cancel after elementary simplification.

Thus, for $n = 2p-1$ odd we find that $LHS = 0$ in equation 91 and thus the generalised root identities at such $\mu = -n$ are immediately validated, without yielding any further new information regarding ζ or its associated quantities.

4.2 The case of $n = 2p$ even

When $n = 2p$ is even we have, from equation 90 in equation 91, that

$$\begin{aligned} LHS &= \left\{ \begin{array}{c} \frac{-1}{2^{2p}} - \frac{1}{2^{2p}} \cdot \left\{ \frac{1}{4(2p+1)} + \frac{1}{2} - \sum_{j=1}^{2p} \binom{2p}{j} \zeta(-j)\, 2^{2j} \right\} \\ + \left\{ \begin{array}{c} -2 \left\{ \frac{4p+1}{2^{2p+3}(2p+1)} \right\} \\ +2^{2p} \cdot \sum_{l=0}^{2p-1} \binom{2p}{l} \frac{B_{2p+1-l}}{(2p+1-l)} \frac{1}{2^{2l}} \\ +2 \cdot (-1)^p \cdot (2p)! \cdot C_S^{(2p)} \end{array} \right\} \end{array} \right\} \\ &= \frac{-1}{2^{2p}} - \frac{1}{2^{2p}} \left\{ \frac{4p+3}{4(2p+1)} + \frac{4p+1}{4(2p+1)} \right\} + 2 \cdot (-1)^p \cdot (2p)! \cdot C_S^{(2p)} \\ &= \frac{-1}{2^{2p-1}} + 2 \cdot (-1)^p \cdot (2p)! \cdot C_S^{(2p)} \end{aligned} \tag{95}$$

where here we have invoked the same relationship between $\zeta(-j)$ and B_{j+1} as in the previous case and changed summation variable in the second sum from l to $j = 2p - l$ in order to engineer the cancellation of the two sums in this expression; and we have performed elementary simplifications on the remaining terms.

Since ζ satisfies the generalised root identities at $\mu = -n = -2p$, so that $LHS = RHS = 0$ in equation 91, we thus deduce the following new result as a consequence of the generalised root identities at such negative even μ-values:

Result 1: *Assuming RH, the argument of the zeta function, $S(t)$ must satisfy the following collection of identities for all $p \in \mathbb{Z}_{>0}$:*

$$C_S^{(2p)} = (-1)^p \frac{1}{(2p)!} \frac{1}{2^{2p}} \tag{96}$$

where $C_S^{(2p)}$ is as defined in subsection 3.3. In light of equations 86 *and* 87, *this is equivalent to the following set of integral identities:*

$$\int_0^\infty t^{2p} \mathrm{d}S(t) = (-1)^p \frac{1}{2^{2p}} \tag{97}$$

which now hold not just for $p \in \mathbb{Z}_{>0}$, but also at $p = 0$. Here, these are strong *Césaro integrals, meaning that*

$$\int_0^T t^{2p} \mathrm{d}S(t) \stackrel{C}{\simeq} (-1)^p \frac{1}{2^{2p}} \qquad \text{as} \quad T \to \infty \tag{98}$$

via the pure power, P^{2p+1}, of the Césaro averaging operator.

Comment: This is a new family of results. It is conditional on RH since assuming RH is necessary for the calculations of section 3, in particular those relating to $S(t)$.

It establishes specific values (at least for n even) for the natural set of strong Césaro integrals, $\int_0^\infty t^n \mathrm{d}S(t)$, which integrate powers of t against the measure $\mathrm{d}S(t)$. For $n \geq 1$, note that we may of course rewrite $\int_0^\infty t^n \mathrm{d}S(t)$ as $\underset{T\to\infty}{Clim} \left\{ T^n S(T) - n \int_0^T t^{n-1} S(t) \mathrm{d}t \right\}$, so that equation 97 can, if desired, be re-expressed as a set of results involving integrals against $S(t)$ itself, rather than against the measure $\mathrm{d}S(t)$.

4.3 Can we go further than Result 1 using the generalised root identities at $\mu = -n$?

We have derived Result 1 in this section by considering the generalised root identities for ζ purely at the single point $s_0 = \frac{1}{2}$. Can any additional results be obtained by considering these identities at other specific s_0-values or general s_0?

The answer is no. The particular case of $s_0 = \frac{1}{2}$ and its associated equations 91 for all $n \in \mathbb{Z}_{\geq 0}$, are in fact equivalent to knowing that the polynomials $r_\zeta(s_0, -n)$ at such n are all identically zero as functions of s_0. Thus no further information is to be gleaned regarding ζ from considering these identities at general $s_0 \neq \frac{1}{2}$.

This is because, if we write the polynomial $r_\zeta(s_0, -n)$ as a polynomial in $(s_0 - \frac{1}{2})$, i.e. as

$$r_\zeta(s_0, -n) = \sum_{j=0}^{n+1} a_j^{(n)} (s_0 - \frac{1}{2})^j \tag{99}$$

then it turns out that

$$a_{n+1}^{(n)} = 0 \tag{100}$$

and that for all $0 \leq j \leq n$ we have

$$a_j^{(n)} = (-1)^j \binom{n}{j} \cdot r_\zeta(\frac{1}{2}, -(n-j)) \quad . \tag{101}$$

Thus the coefficients in this polynomial are all simply multiples of $r_\zeta(\frac{1}{2}, -l)$, $0 \leq l \leq n$. Since a polynomial is identically zero if and only if its coefficients are all zero, we see that $r_\zeta(s_0, -n)$ being identically zero (per the generalised root identity at $\mu = -n$) for all $n \in \mathbb{Z}_{\geq 0}$ is equivalent to simply imposing the conditions that $r_\zeta(\frac{1}{2}, -l) = 0$ for all $l \in \mathbb{Z}_{\geq 0}$, as claimed.

We omit a proof of these claims regarding the $a_j^{(n)}$ here, but suffice to say that Result 1 exhausts the conclusions regarding ζ and $S(t)$ which we can deduce from the generalised root identities at the points $\mu = -n$, $n \in \mathbb{Z}_{\geq 0}$.

The only other point to make in regards to extending Result 1 is very minor and concerns the split between the cases of n odd and n even.

Equation 87 in subsection 3.3 tells us that $\int_0^\infty t^n \mathrm{d}S(t)$ is a well-defined, finite strong-Césaro integral, expressible in terms of $C_S^{(n)}$, irrespective of whether n is odd or even.

When $n = 2p$ is even, the contributions to $r_{NT}(s_0, -2p)$ from roots above and below the real axis reinforce, leaving us a contribution in terms of $C_S^{(2p)}$ within $\overset{\circ}{B}_{-2p}$, and this in turn leads to Result 1, on imposing the generalised root identity at $\mu = -2p$. By contrast, when $n = 2p-1$ is odd, these contributions involving $C_S^{(2p-1)}$ cancel, leaving $\overset{\circ}{B}_{-2p+1} = 0$, and so the generalised root identity at $\mu = -2p+1$ ends up being satisfied without shedding any light on the value of $C_S^{(2p-1)}$ or, by extension, the value of $\int_0^\infty t^{2p-1} \mathrm{d}S(t)$.

Given this, all we can say to at least partially extend to the case of n odd as well as n even, is to extend the integration domain of our strong Césaro integrals from $[0, \infty)$ to $(-\infty, \infty)$. Then, since $S(t)$ is odd in t, we can at least say that $\int_{-\infty}^\infty t^{2p-1} \mathrm{d}S(t)$ is zero (since the integrals on $[0, \infty)$ and $(-\infty, 0]$ are both finite after application of P^{2p} and they are negatives of each other). On the other hand $\int_{-\infty}^\infty t^{2p} \mathrm{d}S(t)$ is double the value of $\int_0^\infty t^{2p} \mathrm{d}S(t)$.

In other words, Result 1 can be extended in a trivial way to at least reflect the implication of oddness of $S(t)$ when n is odd, by rewriting equation 97 to say instead that, for any $n \in \mathbb{Z}_{\geq 0}$

$$\int_{-\infty}^{\infty} t^n \, \mathrm{d}S(t) = \begin{cases} 0 & n = 2p-1 \quad \text{odd} \\ & \\ (-1)^p \frac{1}{2^{2p-1}} & n = 2p \quad \text{even.} \end{cases} \tag{102}$$

4.4 Interpretation and implications of Result 1 - Mellin transforms, Césaro arrays and asymptotic expansions for $\int_0^\infty f(at)\,\mathrm{d}S(t)$ and $\int_{-\infty}^\infty f(at)\,\mathrm{d}S(t)$

The form of equation 97 is clearly reminiscent of Mellin transforms. In our next set of papers, we will show that generalised Césaro convergence is in fact the natural and proper framework under which to consider Mellin transforms. It simultaneously allows an extension of their domain of applicability and simplifies many of the technical conditions which beset their use within the constraints of classical convergence. Within this context, $\int_0^\infty t^n \mathrm{d}S(t) = \int_0^\infty t^n\, S'(t)\,\mathrm{d}t$ is the value of the Mellin transform of $S'(t)$ at $s = n+1$ and equation 97 can be interpreted as evaluating this Mellin transform at the odd positive integers.

Extending this to make sense of $\int_0^\infty t^{s-1}\mathrm{d}S(t)$ at arbitrary s in a generalised Césaro sense - thereby obtaining the full Mellin transform of $S'(t)$ conditional on RH - would of course be very interesting.

However, the difficulty we have encountered for $\mu = s-1$ odd suggests this may be difficult, and for μ arbitrary real or complex the difficulty is compounded both by the fact that repeated integration by parts no longer "ends" after a finite number of steps, and the fact that the derivative side of the generalised root identity at $-\mu$ is not simply 0 as it was for $\mu = n \in \mathbb{Z}_{\geq 0}$, but rather a much more complicated expression involving a sum over primes p. Numerical approximation may nonetheless be feasible and would be interesting to pursue.

An alternative direction in which to extend things based upon either equation 97 or equation 102 is to note that these are both strong Césaro integrals and to harness this within the framework of Césaro arrays developed in [IV]-[VI].

Suppose, for example, that $f(t)$ is an even function with Taylor series in a neighbourhood of 0 given by

$$f(t) = \sum_{j=0}^{\infty} c_{2j} t^{2j} = c_0 + c_2 t^2 + c_4 t^4 + \ldots . \tag{103}$$

Then for u small we still have $f(ut) = \sum_{j=0}^{\infty} c_{2j} t^{2j} u^{2j}$ in some neighbourhood of 0 in t.

If we think of the integral $\int_0^\infty f(ut)\,\mathrm{d}S(t)$ as a "continuous sum" in the "horizontal" t-direction and expand $f(ut)$ via this Taylor series in the "vertical" direction we obtain a generalised Césaro array in which the sum at each height $2p$, representing the coefficient of u^{2p}, is given by $c_{2p} \cdot \int_0^\infty t^{2p}\mathrm{d}S(t)$. Since, by equation 98, $\int_0^T t^{2p}\mathrm{d}S(t)$ is strongly Césaro convergent to $\frac{(-1)^p}{2^{2p}}$ via the pure power P^{2p+1}, we can separate out the value $\frac{(-1)^p}{2^{2p}} \cdot c_{2p}$ at each height and the residual expression is strongly Césaro convergent to zero via P^{2p+1}.

By the general framework of Césaro arrays, we then quarantine these non-zero values at each height and sum them vertically to give us $\sum_{j=0}^{\infty} c_{2j} \frac{(-1)^j}{2^{2j}} u^{2j} = \sum_{j=0}^{\infty} c_{2j} (\frac{iu}{2})^{2j} = f(\frac{iu}{2})$; and the residual expressions at each height combine to give an overall residual term which is Schwartzian in u, i.e. which decays to 0 as $u \to 0$ faster than any power of u and all of whose derivatives behave in the same way.

Thus we conclude in general that under RH we have, for *any* even function $f(t) = \sum_{j=0}^{\infty} c_{2j} t^{2j}$ expandable in a Taylor series near 0, that

$$\int_0^\infty f(ut)\,\mathrm{d}S(t) \;=\; \sum_{j=0}^{\infty} c_{2j} \left(\frac{iu}{2}\right)^{2j} + \mathcal{S}_0(u) \;=\; f(\frac{iu}{2}) + \mathcal{S}_0(u)\,. \tag{104}$$

A corresponding result can be deduced along the same lines for an even function, $f(t)$, with suitable behaviour in a neighbourhood of ∞, this time modulo a Schwartzian component in $\mathcal{S}_\infty(u)$.

As examples of the application of equation 104 we have that, for u small

$$\int_0^\infty \frac{1}{1+u^2t^2}\,\mathrm{d}S(t) = \frac{1}{1-\left(\frac{u}{2}\right)^2} + \mathcal{S}_0(u) \tag{105}$$

and similarly

$$\int_0^\infty \frac{1}{1+u^{2m}t^{2m}}\,\mathrm{d}S(t) = \frac{1}{1+(-1)^m\left(\frac{u}{2}\right)^{2m}} + \mathcal{S}_0(u) \tag{106}$$

for any $m \in \mathbb{Z}_{\geq 1}$. Since the integrals on the LHS are generally classically convergent, it would be interesting to test these claims numerically for very small u (noting that the smaller u is, the more relevant the behaviour of $S(t)$ for large t, and hence the structure of non-trivial zeros on the critical line, becomes).

Indeed, since equation 104 should hold for *arbitrary* f an even function, it would be very interesting to consider what implications might be deduced regarding the behaviour of $S(t)$, and hence the non-trivial roots of ζ, for large t by judicious choice of "testing function" $f(t)$. We shall endeavour to explore this charming and most amiable diversion in a future paper, a prospect which we trust leaves the reader in a state of most delightful anticipation![6]

In a similar vein, but using equation 102 rather than equation 97 as our supporting tool, we have that for any smooth function $f(t)$, with Taylor series given by $f(t) = \sum_{j=0}^{\infty} c_j t^j$ in a neighbourhood of 0, then for u small the associated Césaro array calculations for $\int_{-\infty}^{\infty} f(ut)\,\mathrm{d}S(t)$ give the coefficient of u^{2p-1} as zero and the coefficient of u^{2p} as $2 \cdot \frac{(-1)^p}{2^{2p}} \cdot c_{2p}$. Thus we have

$$\int_{-\infty}^{\infty} f(ut)\,\mathrm{d}S(t) = \left\{ f(\frac{iu}{2}) + f(-\frac{iu}{2}) \right\} + \mathcal{S}_0(u) \tag{107}$$

with a corresponding result for f suitably behaved in a neighbourhood of ∞ and $u \to \infty$.

Thus for example, taking $f(t) = \mathrm{e}^{-it}$ we get the following result:

$$\int_{-\infty}^{\infty} \mathrm{e}^{-itu}\,\mathrm{d}S(t) = \left\{ \mathrm{e}^{\frac{u}{2}} + \mathrm{e}^{-\frac{u}{2}} \right\} + \mathcal{S}_0(u) \tag{108}$$

which gives us an asymptotic expansion for the Fourier transform of $S'(t)$ when u is small.

In the same way as just discussed above, it would be interesting to consider equation 107 for other testing functions, $f(t)$, and to see whether we could explore further the behaviour of $S(t)$ and the non-trivial roots of ζ (conditional on RH) by a clever choice of such testing functions.

[6] Although, if any reader is inspired to undertake such an exploration themselves we heartily encourage them to do so - we do not have any settled results at this stage and would even be glad to join forces in such a ramble!

5 Next steps regarding root identities, generally and for ζ

This brings us to the end of this suite of papers. In the first of them we introduced the concept of generalised root identities for equivalence classes of functions using generalised Césaro methods; and we then proceeded, in the next three papers, to demonstrate their applicability in the case of the Riemann zeta function, culminating in the derivation of the new results of section 4.

Overall, we have seen that generalised root identities provide a new and useful family of constraints on the density and geometric location of zeros and poles of functions, and we have by no means exhausted what could be said regarding them. Nonetheless, it is appropriate to end this set here.

Thus, inspired by the words of Douglas Macarthur (pers. comm.) we "will return" to these issues only later, after we have first turned our attention instead to one last set of new papers, discussing generalised Césaro methods and the notions of Taylor series "to the left", Mellin transforms and applications to integration and asymptotics.

When we do return, these next steps on generalised root identities will branch in two primary directions. First, we will extend their application to ζ. Thus far, we have only applied our generalised geometric Césaro framework to the root side of the root identities for ζ. This has led towards the counting function, $N(T)$, for the non-trivial roots and to our results on $S(t)$. We will instead apply generalised Césaro methods to the *derivative side* of the root identities, based on the formula we obtained for it from the Euler product formula. By doing so, we will connect to the functions, $\pi(X)$, $J(X)$ and $R(X)$ relating to the number of primes less than a given number X, and we will show how to derive directly from Césaro considerations the famous formulae relating them to the generalised roots of ζ. This in turn will open up possibilities for examining these formulae via Césaro arrays in order to better understand their asymptotics, and we will explore these possibilities.

Secondly, having turned our focus so heavily upon ζ, we will return to consider the application of the generalised root identities in many other cases. Just as we saw in [VII] that the generalised root identities for $\cos\left(\frac{\pi z}{2}\right)$ are equivalent to the functional equation for ζ, we will show that these root identities lead to many other interesting identities when applied to other well-known functions; and we will consider also the possibilities of chaining such applications together in sequence.

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