Taylor Series to the Left I - Integration

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April 18, 2026

Abstract

$$\begin{gathered} \textit{Like the creeper that girdles the tree trunk} \\ \textit{It winds around, subtle and deft;} \\ \textit{For its terms march away to the rightwards} \\ \textit{But they also creep off to the left!} \end{gathered}$$

This is the first in a set of papers concerning basic power series, starting with the subject of the above poem, the Taylor-McLaurin series. We introduce the notion of "Taylor series to the left" and use it to derive new approaches and, surprisingly, new results in many elementary areas in mathematics. These include elementary integration, the asymptotic behaviour of functions, local-to-global inference, Mellin transforms, and calculation involving formal symbols. In all cases, we show that the natural context for these ideas involves their extension from the realm of classical convergence to the generalised geometric Césaro framework, with associated simplifications and insights. In this first paper we introduce the concept of Taylor series to the left heuristically and we consider many examples showing their initial application to the calculation of definite integrals of the form $\int_0^\infty f(x)\,\mathrm{d}x$ and, in turn, to other related definite integrals. While the discussion begins heuristically, we end up deriving results which make it well-founded. As a natural part of this work we begin discussion of key related questions - namely gauge-freedom and gauge-invariance; local-to-global inference and asymptotic behaviour of functions; and generalised Césaro extension and the associated concepts of Taylor/Mellin transform. All of these will become central concerns in the succeeding papers.

1 Introductory comments regarding this set of papers

The topics considered in this set of papers are all elementary. Indeed they start from a high-school level treatment of Taylor-McLaurin series and for the most part move only to aspects of one-variable complex analysis, power series and Mellin transforms which are still generally introductory - albeit that there are occasional forays into areas (for example, work related to Ramanujan's so-called "master theorem") which are more specialised.

Nonetheless, despite this focus on basics, we emphasise at the outset that we believe every paper in the set contains genuinely new results - either new perspectives on long-established results; or new ways of viewing and applying existing results; or extensions of existing results to new settings; or completely new results altogether!

In light of this unusual mixture of elementary and new; of basic ideas and alternative perspectives; and of familiar and novel - we start our discussion of each new topic by adopting a perspective of "moral mathematics". That is to say, we try first to explain what morally ought to be true, and what the heuristic motivation of each idea is, and then move next to exploring examples as a way to test and refine these ideas.[1] Nonetheless, over the course of the full set of papers, we do resolve these initially loose ideas into tighter propositions and results, supported by rigorous proofs.

1.1 Detailed introduction to this paper

In this first paper in the series, we introduce the notion of "Taylor series to the left" as a natural extension of the usual Taylor series of a function, $f(x)$, at $x = 0$.[2]

Doing so naturally entails embedding the Taylor-coefficient set within a continuous "fabric" - in other words, as the values at the non-negative integer points of a Taylor-coefficient *function*, $\overset{\vee}{f}(s)$, of a continuous variable s - and hence extending leftwards to negative integer values.

We explain heuristically why the first new such Taylor coefficient on the left should relate to $\int_0^\infty f(x)\,\mathrm{d}x$, and why further-left coefficients should in turn relate to higher iterated integrals of f. By working in moral terms - in a way that takes seriously the coefficient function at general s, not just at integer values - we then derive an explicit formula giving $\int_0^\infty f(x)\,\mathrm{d}x$ in terms of the single derivative value $\overset{\vee}{f}'(-1)$.

Of course, going from the known Taylor coefficient-set at $n \in \mathbb{Z}_{\geq 0}$, to a Taylor coefficient function of a continuous variable $s \in \mathbb{R}$ (or indeed $s \in \mathbb{C}$) is not a uniquely well-defined process. There is what physicists would characterise as gauge-freedom in doing so. Thus questions of identifying a canonical choice and of guaranteeing the gauge-invariance of quantities like $\int_0^\infty f(x)\,\mathrm{d}x$, independent of the gauge-choice of coefficient function, become relevant.

In many cases, however, there is a relatively natural way of defining $\overset{\vee}{f}(s)$, and the resulting approach to the calculation of $\int_0^\infty f(x)\,\mathrm{d}x$ as merely the derivative

[1] This is consistent with our preference in all our previous sets of papers for prioritising elegance and utility over dry theorem-lemma-proof exposition, and for ensuring we do not get bogged down by technical rigour and formalism too early in the development of ideas and methods.

[2] Perhaps the terminology should refer to McLaurin rather than Taylor, but for the author the die is cast and it is too late to retreat back across this terminological Rubicon!

value $\overset{\vee}{f}'(-1)$ often seems (at least to the author) both unexpected and surprisingly simple - a way of calculating an integral without using anything resembling traditional integration techniques! We illustrate this with a variety of examples, and we use some minor stumbles along the way to explore these gauge-questions and to refine our pursuit of a canonical approach to passing from coefficient set to fabric.

Additionally, some of the later examples point towards a number of extensions. First, they indicate that the relationship between $\int_0^\infty f(x)\,\mathrm{d}x$ and $\overset{\vee}{f}'(-1)$ continues to hold in a generalised Césaro setting, where $\int_0^\infty f(x)\,\mathrm{d}x$ is not classically well-defined but does exist as a generalised Césaro integral.

Secondly they indicate that this relationship continues to hold even where the radius of convergence of the Taylor series near 0 is itself zero - so that this series is a power series giving the *asymptotic* expansion for f near 0, but is not actually convergent at any x.

To go further, we then consider the behaviour of coefficient functions under elementary operations, in particular the multiplication of the original function by a power of x. This leads directly to the question of evaluating Mellin transforms, $\mathcal{M}[f](s) := \int_0^\infty x^{s-1} f(x)\,\mathrm{d}x$.

By using the progress made in our examples, we show that we can in fact bootstrap from our relationship between $\int_0^\infty f(x)\,\mathrm{d}x$ and $\overset{\vee}{f}'(-1)$, to a general formula relating the Mellin transform of f, $\mathcal{M}[f](s)$, and its Taylor coefficient function, $\overset{\vee}{f}(s)$, for arbitrary s. Inverting this relationship in turn gives us the rigorous canonical way of defining $\overset{\vee}{f}(s)$ which we have been seeking - in terms of the Mellin transform. Under this canonical choice, the gauge-questions which have arisen previously, and their associated questions regarding sign-issues in evaluating gauge-invariant quantities like $\int_0^\infty f(x)\,\mathrm{d}x$, are all resolved.

We conclude with some reflections on remaining open issues and some straightforward extensions that allow these simple techniques (of what might be called "integrating by differentiating without using any normal integration techniques") to be extended from the case of $\int_0^\infty f(x)\,\mathrm{d}x$ to other definite integrals, like $\int_a^\infty f(x)\,\mathrm{d}x$ and $\int_a^b f(x)\,\mathrm{d}x$ for arbitrary a and b.

1.2 Brief roadmap to the subsequent papers within this set

Sketching now very briefly what is covered in the remaining papers in this set, the second paper ([XII]) is short. It explores another aspect of Taylor series to the left - namely their relationship to the asymptotic behaviour of f near ∞. We consider various examples, thereby formulate a precise claim, and discuss the surprising local-to-global implications of this claim.

In the third paper ([XIII]) we then consider how to place the content of these first two papers within the unifying context of a natural "Taylor transform". We show that this can be done, provided we move to work within the generalised

Césaro framework. There we are able to introduce a generalised Césaro inner product from which the desired Taylor transform arises naturally.

But it is then readily shown that this Taylor transform is essentially just the familiar Mellin transform provided f is sufficiently well-behaved.

It follows from all this that generalised Césaro convergence, not classical convergence, is in fact the right framework within which to apply the Mellin transform. We show that, when this is done, not only is its domain of applicability greatly increased, but many of the technicalities which beset the traditional treatment of Mellin transforms - such as those regarding s-strip restrictions on its inversion - disappear.

There are, moreover, other benefits. The calculation of Mellin transforms by the sort of bootstrapping methods introduced in this paper ([XI]) become possible, and we further extend such methods by stretching our storehouse of techniques for directly constructing the canonical form of $\overset{\vee}{f}(s)$.

The connection of Taylor-coefficient functions and Mellin transforms to power series expansions for f near 0 and ∞ also becomes natural and easy to intuit. Indeed, we are readily able to convert our conjectural claim from [XII] into a firm result relating $\overset{\vee}{f}(s)$ at integer-values $s = m$ to the difference between the coefficient of x^m in the power series for f near $x = 0$ and its coefficient in the power series for f near $x = \infty$. As we discuss at the end of this paper, this is very closely related to Ramanujan's so-called "master-theorem", but both places it in a broader generalised Césaro context and resolves it rigorously into a single result *simultaneously* encapsulating the behaviour of f near $x = 0$ and its asymptotic behaviour as $x \to \infty$.

The fourth paper in the set ([XIV]) is then very short and shows the capacity of these results on Taylor series to the left to facilitate radical local-to-global deduction of asymptotic behaviour as $x \to \infty$ purely from local examination of a function in an infinitesimally small neighbourhood of 0 - that is, to see to the edge of the universe with a microscope! It does so in a clearly interesting and non-trivial instance - and one where the extension to the generalised Césaro framework is essential.

Finally, the fifth and last paper in the set ([XV]) considers a miscellaneous assortment of further applications of these ideas.

First we consider the application of the Taylor-series-to-the-left theory we have developed to the case where the underlying function is a p-sum function. Such cases are of great interest in analytic number theory in general and in the theory of the Riemann zeta function in particular, and we start with an example which shows the depth of the connection of these ideas to generalised Césaro theory and which emphatically confirms the correctness of viewing the generalised Césaro framework as the right framework within which to consider them. We then consider other examples which show the utility of Taylor-series-to-the-left methods in carrying out such p-sum Mellin transform calculations, and illustrate the new implications which they give rise to about the asymptotic behaviour of these p-sums.

Next we show additional examples evaluating certain definite integrals by

Taylor-series-to-the-left methods, which would otherwise be very difficult to calculate by traditional methods. These include a wholly new result (at least one the author can't find in Gradshteyn and Ryzhik! [GR]) evaluating $\int_0^\infty \mathrm{e}^{-p(x)}\,\mathrm{d}x$ for an arbitrary polynomial $p(x)$ of any degree with positive leading coefficient.

After this we consider Césaro extension to integrals with poles on the contour of integration, and connect different possible contour-amended evaluations of these to different gauge-choices. Lest we become overly complacent, however, we also discuss limitations which remain, and which mean that the use of Taylor series to the left does not act as a panacea - giving examples to show that $\int_0^\infty f(x)\,\mathrm{d}x$ still remains intractable even for some very simple and natural functions f.

Finally, we consider the behaviour of TLA-coefficient functions under complex underlying operations - the search for a so-called "calculus" of TLA-coefficient functions - and in particular whether progress can be made by utilising the concepts of formal symbols and formal function elements introduced in [III] and applying combinatorial and algebraic methods. In general we find that, while some progress can be made, it is far from straightforward and involves further subtleties regarding the order of combination of such symbols - and how this relates to the interpretation of power series asymptotics near 0 and near ∞ - which are both challenging and interesting.

We conclude with a brief collection of wild speculations regarding issues of interest and next avenues to explore.

1.3 Final notes

As the poem at the start of the abstract hints, throughout this set of papers there will be occasional reference to the work of a variety of Kipling scholars from the KIM (Kafiristan Institute of Mathematics) with whom the author has had the honour to collaborate. In particular, the author appreciates many valuable personal communications from such scholars as Peachy Carnahan and Daniel Dravot; Purun Das; Danny Deever; Morowbie Jukes and myriad others.

2 Taylor series to the left

Suppose $f : \mathbb{R} \to \mathbb{R}$ is a smooth function, in all respects as well-behaved as may be needed, and in particular decaying sufficiently fast as $x \to \infty$ that $\int_0^\infty f(x)\,\mathrm{d}x$ is classically well-defined. Then the Taylor series for f near $x = 0$ is given by

$$\sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} x^j = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots \tag{1}$$

with some radius of convergence, R, and for $-R < x < R$ this Taylor series is classically convergent to $f(x)$. In this formulation the Taylor series indices $j = 0, 1, 2, \dots$ extend only off in the rightwards direction.

If we were to formally extend left to $j = -1, -2, \ldots$ we would get the additional terms

$$\frac{f^{(-1)}(0)}{(-1)!}\frac{1}{x} + \frac{f^{(-2)}(0)}{(-2)!}\frac{1}{x^2} + \frac{f^{(-3)}(0)}{(-3)!}\frac{1}{x^3} + \ldots \quad . \tag{2}$$

These make up what we shall call loosely the "Taylor series to the left" for f at 0. The first order of business is to make sense of what these terms might mean.

Well, since each increment by 1 in j for $j \geq 0$ corresponds to taking one extra derivative of f before evaluating at 0, so each decrement by 1 in j for $j < 0$ should correspond to taking one further anti-derivative of f before evaluating at 0. Since $F(x) := \int_x^\infty f(t)\,\mathrm{d}t$ is the negative of an anti-derivative of f, and since this is the only natural definition of such an anti-derivative which doesn't involve making an arbitrary choice of finite integration limit, it follows that morally, $f^{(-1)}(0)$ should be given by

$$f^{(-1)}(0) := -\int_0^\infty f(x)\,\mathrm{d}x\,. \tag{3}$$

In the same way, if we let $\{F_n(x)\}_{n=0}^\infty$ be the sequence of iterated indefinite integrals given by

$$F_0(x) := f(x) \qquad \text{and} \qquad F_{n+1}(x) := \int_x^\infty F_n(t)\,\mathrm{d}t \quad \text{for all } n \in \mathbb{Z}_{\geq 0} \tag{4}$$

then we should have

$$f^{(-n)}(x) = (-1)^n F_n(x) \quad \text{and thus} \quad f^{(-n)}(0) = (-1)^n F_n(0) \quad \text{for all } n \in \mathbb{Z}_{\geq 0}\,. \tag{5}$$

As for the denominators in the terms in our Taylor series to the left, the factorial function $s!$ has simple poles at each of the negative integer points, so for all $n \in \mathbb{Z}_{>0}$ we have $(-n)! = \pm\infty$. As a result, every coefficient $\frac{f^{(-n)}(0)}{(-n)!}$ in the Taylor series to the left is, in fact, zero! What to make of this?

Well, looking on the bright side, it's good news that including these extra terms will not disturb the existing Taylor series expression, which already correctly converges to $f(x)$ for $-R < x < R$. On the other hand, however, what is the point of studying Taylor series to the left if they are always identically zero?

The answer is that, properly viewed, they are *not* just a discrete set of identically zero terms. Specifically, let us extend the Taylor coefficients, $\frac{f^{(j)}(0)}{j!}$, from being just a sequence indexed by $j \in \mathbb{Z}$ to a function, which we denote by $\overset{\vee}{f}(s)$, defined for all $s \in \mathbb{R}$ and coinciding with the given values for $s = j \in \mathbb{Z}$. We call $\overset{\vee}{f}(s)$ the Taylor coefficient *function* of f and it should then, morally, be given by something like

$$\overset{\vee}{f}(s) = \frac{\left(\frac{d}{dx}\right)^s [f](0)}{s!}\,. \tag{6}$$

Viewed this way, we still have $\overset{\vee}{f}(-1) = 0 = \overset{\vee}{f}(-2) = \overset{\vee}{f}(-3) = \ldots$, but the behaviour of the Taylor coefficient function in the vicinity of the negative integer points now becomes interesting! We could try to investigate this behaviour based on equation 6 using fractional calculus or Fourier theory to try to make sense of the numerator (as we did for root identities in the previous suite of papers), but instead it will be better to follow our nose and to work for now in a much more practical (and moral) fashion.

Assuming only that the numerator in equation 6 varies smoothly in s, then for $s = -1 + \epsilon$, ϵ small, we have that the numerator in $\overset{\vee}{f}(-1+\epsilon)$ is given by $f^{(-1)}(0) + O(\epsilon) = -\int_0^\infty f(x)\,\mathrm{d}x + O(\epsilon)$; while the denominator is given by $(-1+\epsilon)! = \frac{1}{\epsilon} + const + O(\epsilon)$ and thus $\frac{1}{(-1+\epsilon)!} = \epsilon + O(\epsilon^2)$. It follows that

$$\overset{\vee}{f}(-1+\epsilon) = -\left\{\int_0^\infty f(x)\,\mathrm{d}x\right\}\epsilon + O(\epsilon^2)$$

and since $\overset{\vee}{f}(-1) = 0$, so

$$\overset{\vee}{f}{}'(-1) = \lim_{\epsilon\to 0}\frac{\overset{\vee}{f}(-1+\epsilon)}{\epsilon} = -\int_0^\infty f(x)\,\mathrm{d}x\,. \tag{7}$$

Thus we have the following conjecture:

Conjecture 1: *Suppose f is smooth and decays sufficiently fast as $x \to \infty$ to be classically integrable on $[0,\infty)$. Then its Taylor coefficient function $\overset{\vee}{f}(s)$ satisfies $\overset{\vee}{f}(-1) = 0$ and we have*

$$\int_0^\infty f(x)\,\mathrm{d}x = -\overset{\vee}{f}{}'(-1) = -\lim_{\epsilon\to 0}\frac{\overset{\vee}{f}(-1+\epsilon)}{\epsilon}\,. \tag{8}$$

This conjecture can be extended to the left to the higher iterated integrals of f. For any $n \in \mathbb{Z}_{>0}$, let $g(x) = F_n(x) = \frac{1}{(-1)^n}f^{(-n)}(x)$ and assume that f is smooth and decays sufficiently fast that $\int_0^\infty F_n(x)\,\mathrm{d}x$ is classically well-defined, so that $\overset{\vee}{g}(-1) = \overset{\vee}{f}(-n-1) = 0$. Then, since $\frac{1}{(-n-1+\epsilon)!} = (-1)^n \cdot n! \cdot \frac{1}{(-1+\epsilon)!} + O(\epsilon^2)$, it follows that

$$\begin{aligned}
\int_0^\infty F_n(x)\,\mathrm{d}x &= \int_0^\infty g(x)\,\mathrm{d}x = -\lim_{\epsilon\to 0}\frac{\overset{\vee}{g}(-1+\epsilon)}{\epsilon} \\
&= -\lim_{\epsilon\to 0}\frac{1}{\epsilon}\cdot\left\{\frac{\left(\frac{d}{dx}\right)^{-1+\epsilon}[g](0)}{(-1+\epsilon)!}\right\} \\
&= -\lim_{\epsilon\to 0}\frac{1}{\epsilon}\cdot\frac{1}{(-1)^n}\cdot\left\{\frac{\left(\frac{d}{dx}\right)^{-n-1+\epsilon}[f](0)}{(-n-1+\epsilon)!}\right\}\cdot\frac{1}{(-1)^n n!} \\
&= -\frac{1}{n!}\cdot\lim_{\epsilon\to 0}\frac{\overset{\vee}{f}(-n-1+\epsilon)}{\epsilon} = -\frac{1}{n!}\cdot\overset{\vee}{f}{}'(-n-1)\,.
\end{aligned}$$

Thus the above conjecture can be extended as

Conjecture 2: *Suppose f is smooth and decays sufficiently fast as $x \to \infty$ that all of $f(x), F_1(x), \dots, F_n(x)$ are classically integrable on $[0, \infty)$. Then for all $0 \le j \le n$ we have $\overset{\vee}{f}(-j-1) = 0$ and*

$$\int_0^\infty F_j(x)\,\mathrm{d}x = -\frac{1}{j!}\,\overset{\vee}{f}'(-j-1) = -\frac{1}{j!}\lim_{\epsilon \to 0} \frac{\overset{\vee}{f}(-j-1+\epsilon)}{\epsilon}. \qquad (9)$$

Alternatively, rather than considering iterated integrals of f, we could instead extend to the left by considering functions of the form $g_n(x) := x^n f(x)$. Then immediately we should have $\overset{\vee}{g}_n(s) = \overset{\vee}{f}(s-n)$ and conjecture 1 implies

Conjecture 3: *Suppose f is smooth and decays sufficiently fast as $x \to \infty$ that all of $f(x), g_1(x), \dots, g_n(x)$ are classically integrable on $[0, \infty)$. Then for all $0 \le j \le n$ we have $\overset{\vee}{f}(-j-1) = 0$ and*

$$\int_0^\infty g_j(x)\,\mathrm{d}x = \int_0^\infty x^j f(x)\,\mathrm{d}x = -\overset{\vee}{f}'(-j-1) = -\lim_{\epsilon \to 0} \frac{\overset{\vee}{f}(-j-1+\epsilon)}{\epsilon}. \qquad (10)$$

Comments: **(i)** Under the decay conditions imposed in both, conjectures 2 and 3 are equivalent, since repeated integration by parts readily converts $\int_0^\infty x^j f(x)\,\mathrm{d}x$ into the corresponding integral $\frac{1}{j!}\int_0^\infty F_j(x)\,\mathrm{d}x$ and vice-versa.

(ii) Conjectures 1-3 are at this stage very loose because we have not clarified in any detail how to pass from the discrete Taylor coefficient set $\frac{f^{(j)}(0)}{j!}$ to the continuous coefficient *function* $\overset{\vee}{f}(s)$, $s \in \mathbb{R}$.

As noted, we could try to work from equation 6 via fractional calculus or Fourier methods, but this turns out not to be the best way. Instead we look to find a *natural* way to perform the extension, and in so doing to find a more familiar way of defining $\overset{\vee}{f}(s)$ canonically.

Approached this way, the problem of extension has what physicists would recognise as the issue of gauge-freedom. For any given choice of extension, $\overset{\vee}{f}(s)$, a seemingly equally good choice would be $\overset{\vee}{f}(s) \cdot \phi(s)$ where $\phi(s)$ is any smooth function satisfying $\phi(j) = 1$ for all $j \in \mathbb{Z}$ (such as, for example, $\phi(s) = \cos(2\pi s)$). Here $\phi(s)$ would be considered a gauge-transformation which leaves the values of the extension-function unaltered at the integer points where the Taylor coefficient-set is already prescribed.

The task is to find, among all the possible gauge-equivalent choices, a canonical "right" choice of Taylor coefficient extension function to take as $\overset{\vee}{f}(s)$, consistent with conjectures 1-3. Here we are noting that results like formulae 8-10 in conjectures 1-3 are not gauge-independent, since quantities like $\int_0^\infty f(x)\,\mathrm{d}x$ are well-defined independent of the gauge-choice of $\overset{\vee}{f}$, but the derivatives on the RHS in these formulae are dependent on the gauge.

To find the "right" way to obtain a canonical Taylor coefficient function extension under which conjectures 1-3 will become theorems, we now focus on conjecture 1 (since the others follow from it) and consider a number of simple examples.

3 Example calculations

Example 1(a) [$f(x) = \mathbf{e}^{-x}$]: Here the Taylor series around 0, with $R = \infty$, is $\mathrm{e}^{-x} = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} x^j$, so $\overset{\vee}{f}(m) = \frac{(-1)^m}{m!}$ for $m \in \mathbb{Z}$ and it seems natural to take $\overset{\vee}{f}(s) = \frac{\cos(\pi s)}{s!}$. Then

$$\lim_{\epsilon \to 0} \frac{\overset{\vee}{f}(-1+\epsilon)}{\epsilon} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \frac{\cos(\pi(-1+\epsilon))}{(-1+\epsilon)!} = -1$$

and since $\int_0^\infty \mathrm{e}^{-x}\,\mathrm{d}x = 1$ it follows that we do have $\int_0^\infty f(x)\,\mathrm{d}x = -\lim_{\epsilon \to 0} \frac{\overset{\vee}{f}(-1+\epsilon)}{\epsilon}$ per conjecture 1. So far, so good!

Example 1(b) [$f(x) = \mathbf{e}^{-x^2}$]: Here the Taylor series around 0, with $R = \infty$, is $\mathrm{e}^{-x^2} = 1 - x^2 + \frac{1}{2!}x^4 - \frac{1}{3!}x^6 + \ldots$, so $\overset{\vee}{f}(m) = \cos(\pi \frac{m}{2}) \frac{1}{(\frac{m}{2})!}$ for $m \in \mathbb{Z}$ and it thus seems natural to take $\overset{\vee}{f}(s) = \cos(\pi \frac{s}{2}) \frac{1}{(\frac{s}{2})!}$. This time the fact that $\overset{\vee}{f}(-1) = 0$ arises not from the factorial in the denominator, but from the factor of $\cos(\pi \frac{s}{2})$ in the numerator, and we have

$$\lim_{\epsilon \to 0} \frac{\overset{\vee}{f}(-1+\epsilon)}{\epsilon} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \cdot \cos(\frac{\pi}{2}(-1+\epsilon)) \frac{1}{(-\frac{1}{2})!} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \frac{\pi\epsilon}{2} \frac{1}{\sqrt{\pi}} = \frac{\sqrt{\pi}}{2}$$

on noting that $(-\frac{1}{2})! = \Gamma(\frac{1}{2}) = \sqrt{\pi}$. On the other hand, we know that $\int_0^\infty \mathrm{e}^{-x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}$. It follows that in this case we have $\int_0^\infty f(x)\,\mathrm{d}x = \lim_{\epsilon \to 0} \frac{\overset{\vee}{f}(-1+\epsilon)}{\epsilon}$ and so, under this gauge-choice of $\overset{\vee}{f}$, conjecture 1 fails but only by a sign-factor of -1. Let us press on!

Example 1(c) [$f(x) = \mathbf{e}^{-x^n}$, $n \in \mathbb{Z}_{\geq 1}$ arbitrary]: Here the Taylor series around 0, with $R = \infty$, is $\mathrm{e}^{-x^n} = 1 - x^n + \frac{1}{2!}x^{2n} - \frac{1}{3!}x^{3n} + \ldots$. To get a formula for $\overset{\vee}{f}(s)$ we need a function which will alternate between 1 and -1 with a spacing of n, and with value zero at all integer points off the lattice $\{0, \pm n, \pm 2n, \ldots\}$. There are many ways we could do this, but the cleanest is by taking a ratio of sine-functions as follows:

Lemma 1a: *For any $n \in \mathbb{Z}_{\geq 1}$ the function $\frac{1}{2n} \cdot \frac{\sin(2\pi s)}{\sin(\frac{\pi s}{n})}$ is smooth. It has value 1 at all points $s = 2jn$ ($j \in \mathbb{Z}$); has value -1 at all points $s = (2j+1)n$ ($j \in \mathbb{Z}$); and has value 0 at all integer points not on the lattice $\{0, \pm n, \pm 2n, \ldots\}$.*

This is easy to see. The sine-function on the numerator is zero at all integer points and the ratio is therefore only non-zero at such points when the sine-function on the denominator is also zero, which occurs at the points on the lattice mentioned. At such points the limiting value alternates between $\pm 2n$ based on a L'Hopital's calculation, and the overall factor of $\frac{1}{2n}$ reduces this to alternation between ± 1 as desired.

Using this in our example, the natural formula to take for the Taylor coefficient function of $f(x) = \mathrm{e}^{-x^n}$ is $\overset{\vee}{f}(s) = \frac{1}{2n} \cdot \frac{\sin(2\pi s)}{\sin(\frac{\pi s}{n})} \cdot \frac{1}{(\frac{s}{n})!}$. Then clearly $\overset{\vee}{f}(-1) = 0$ and for any $n > 1$ we have that

$$\lim_{\epsilon \to 0} \frac{\overset{\vee}{f}(-1+\epsilon)}{\epsilon} = \frac{1}{(\frac{-1}{n})!} \cdot \frac{-1}{\sin(\frac{\pi}{n})} \cdot \lim_{\epsilon \to 0} \frac{\sin(2\pi(-1+\epsilon))}{2n\epsilon} = -\frac{\pi}{n} \cdot \frac{1}{(\frac{-1}{n})!} \cdot \frac{1}{\sin(\frac{\pi}{n})} .$$

Since $z! = \Gamma(z+1)$ satisfies the relationship $\Gamma(z+1) = z\Gamma(z)$ and the functional equation $\frac{\pi}{\sin(\pi z)} = \Gamma(z) \cdot \Gamma(1-z)$, this simplifies readily to give that

$$\lim_{\epsilon \to 0} \frac{\overset{\vee}{f}(-1+\epsilon)}{\epsilon} = -\left(\frac{1}{n}\right)! \quad .$$

On the other hand, we know from [1, identity 3.326(1), p336] that $\int_0^\infty \mathrm{e}^{-x^n}\, \mathrm{d}x = (\frac{1}{n})!$. It follows that under our gauge-choice here of $\overset{\vee}{f}$ using Lemma 1a, conjecture 1 is satisfied for $f(x) = \mathrm{e}^{-x^n}$ for any $n \in \mathbb{Z}_{>1}$.

Comments: (i) In examples 1(a)-(c) we have calculated $\overset{\vee}{f}'(-1)$ and then compared it with the known value of $\int_0^\infty f(x)\, \mathrm{d}x$ in order to test the validity of conjecture 1 and investigate the right canonical form for $\overset{\vee}{f}$. It is worth stopping, however, to reflect that the integral in 1(b) is non-trivial, and the integrals in 1(c) for general n are even more so. As such, modulo getting this canonical form nailed down, this approach and equation 8 in conjecture 1 represent a surprising new way of attacking definite integrals of the form $\int_0^\infty f(x)\, \mathrm{d}x$ - one which appears to calculate such integrals without ever using anything resembling traditional integration techniques. In future papers, we will both extend the scope of methods for such "integration without integrating" and use them to calculate some definite integrals of this form which have previously defied calculation (at least as far as we can tell from a perusal of [1]!).

(ii) Example 1(c) subsumes example 1(b), which corresponds to the case of $n = 2$. Our previous naive treatment of 1(b), however, led to a sign-error violation of conjecture 1, whereas there is no sign-error regarding conjecture 1 in 1(c) - for this case or indeed any other n. What went wrong in our earlier treatment of $f(x) = \mathrm{e}^{-x^2}$ in example 1(b)?

Well, for $n = 2$ the formula developed in 1(c) leads to the formula $\overset{\vee}{f}(s) = \cos(\pi s)\cos(\pi \frac{s}{2})\frac{1}{(\frac{s}{2})!}$, which agrees with the formula from 1(b) but has an extra factor of $\cos(\pi s)$.

This extra factor has the value 1 whenever s is an even integer, so it represents a gauge-choice that leaves the even-index Taylor-coefficients unaffected and does not affect the odd-index Taylor-coefficients simply because they are all zero. It does, however, affect the *derivative* of $\overset{\vee}{f}(s)$ at the odd-index $s = -1$. There it supplies the extra factor of -1 that brings example 1(b) back into agreement with conjecture 1, removing the previous sign-error.

(iii) One of the reasons example 1(c) gave the "right" form of $\overset{\vee}{f}(s)$ where example 1(b) did not, is because the function defined in lemma 1 behaves correctly to capture re-scaling $x \to x^n$ in the argument of a function.

By contrast, simply going from $\cos(\pi s)$ (our choice to capture the alternating sign of coefficients when $n = 1$ in example 1(a)) to $\cos(\pi \frac{s}{n})$ - as we did for $n = 2$ in example 1(b) - no longer works even for the original Taylor-coefficient set when $n \geq 3$; and fails by a sign-error for conjecture 1 even for $n = 2$.

This suggests that ensuring our choices in constructing $\overset{\vee}{f}(s)$ all re-scale in a manner consistent with conjecture 1 is a fundamental constraint which needs to be met in our efforts to find a canonical form for $\overset{\vee}{f}(s)$.

To this end, the following lemma - which mirrors lemma 1a but without the alternation of sign - should, morally, undergird the "right" way to go from the Taylor-coefficient function of an arbitrary function $f(x)$ to the Taylor-coefficient function of $g(x) := f(x^n)$:

Lemma 1b: *For any $n \in \mathbb{Z}_{\geq 1}$ the function $\frac{1}{n} \cdot \frac{\sin(2\pi s)}{\sin(\frac{2\pi s}{n})}$ is smooth and has the value 1 at all points $s = jn$ ($j \in \mathbb{Z}$); and has value 0 at all integer points not on the lattice $\{0, \pm n, \pm 2n, \ldots\}$. If $f(x)$ has Taylor-coefficient function $\overset{\vee}{f}(s)$ under which conjecture 1 is satisfied, and if $g(x) := f(x^n)$, then the correct canonical form of $\overset{\vee}{g}(s)$ should be*

$$\overset{\vee}{g}(s) := \frac{1}{n} \cdot \frac{\sin(2\pi s)}{\sin(\frac{2\pi s}{n})} \cdot \overset{\vee}{f}(\frac{s}{n}) \quad , \tag{11}$$

meaning that under this gauge-choice of $\overset{\vee}{g}(s)$, conjecture 1 is also satisfied for $g(x)$.

Note that this lemma is consistent with lemma 1a in the sense that we end up with the same Taylor-coefficient function for $g(x) := f(x^n)$ when the Taylor series for f is itself alternating with period 1, irrespective of whether we apply lemma 1a directly or absorb the alternation within the definition of $\check{f}(s)$ and apply lemma 1b.

We return to this lemma shortly in order to consider Mellin transforms in general and thereby derive a final canonical form for the Taylor-coefficient function of an arbitrary function f. But for now, we defer this and heed instead the advice of Professors Peachy Carnahan and Daniel Dravot [pers. comm.] to the effect that:

The wind is in the palm trees
And the temple bells they say
Oh get you back to more examples
get you back without delay!

Example 2(a) [$f(x) = \frac{1}{1+x}$]: Here the Taylor series around 0, with $R = 1$, is $\frac{1}{1+x} = \sum_{j=0}^{\infty}(-1)^j x^j$, so $\check{f}(m) = (-1)^m$ for $m \in \mathbb{Z}$ and it seems natural to take $\check{f}(s) = \cos(\pi s)$. Then $\check{f}(-1) = -1 \neq 0$. This is good, and consistent with conjecture 1 - it reflects the fact that $f(x) = \frac{1}{1+x}$ is not classically integrable on $[0, \infty)$ since it decays like $\frac{1}{x}$ as $x \to \infty$. We shall return to consider the subtler meaning of this example in a future paper in this series.

Example 2(b) [$f(x) = \frac{1}{1+x^2}$]: Here the Taylor series around 0, with $R = 1$, is $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots$, so using lemma 1a and adopting the corrected general form settled on in example 1(c) we take $\check{f}(s) = \cos(\pi s)\cos(\pi\frac{s}{2})$. Thus $\check{f}(-1) = 0$ and

$$\check{f}'(-1) = (-1)\cdot\lim_{\epsilon\to 0}\frac{\cos(\pi\frac{(-1+\epsilon)}{2})}{\epsilon} = -\frac{\pi}{2}\,.$$

Since $\int_0^\infty \frac{1}{1+x^2}\,\mathrm{d}x = [tan^{-1}(x)]_0^\infty = \frac{\pi}{2}$, conjecture 1 continues to hold under this choice of $\check{f}$ and prospects continue fair (as well as being another nice example of integrating without integrating!).

Example 2(c) [$f(x) = \frac{1}{1+x^n}$, $n \in \mathbb{Z}_{>2}$ arbitrary]: Here the Taylor series around 0, with $R = 1$, is $\frac{1}{1+x^n} = 1 - x^n + x^{2n} - x^{3n} + \ldots$, and lemma 1a gives $\check{f}(s) = \frac{1}{2n}\cdot\frac{\sin(2\pi s)}{\sin(\frac{\pi s}{n})}$. Thus $\check{f}(-1) = 0$ and

$$\check{f}'(-1) = \frac{(-1)}{2n\sin(\frac{\pi}{n})}\cdot\lim_{\epsilon\to 0}\frac{\sin(2\pi(-1+\epsilon))}{\epsilon} = -\frac{\frac{\pi}{n}}{\sin(\frac{\pi}{n})} = -(\frac{1}{n})!(-\frac{1}{n})! \quad .$$

Again, as in example 2(b), this can be taken either as a confirmation of conjecture 1 for this choice of $\check{f}$ by noting (from [1, 3.241(2), p322]) that $\int_0^\infty \frac{1}{1+x^n}\,\mathrm{d}x =$

$\frac{\pi}{n}\csc(\frac{\pi}{n}) = (\frac{1}{n})!(-\frac{1}{n})!$; or else it can be taken as another nice example of efficiently computing a non-trivial definite integral via Taylor-series-to-the-left methods without using anything like contour integration, partial fractions or other traditional techniques of integration.

Comment: Note that in the examples 2(a)-(c) we have correctly calculated $\int_0^\infty f(x)\,\mathrm{d}x$ even though the Taylor series only has radius of convergence 1. Taylor-series-to-the-left methodology does not require that the Taylor series converge on the whole of the domain of integration $[0,\infty)$.

With lemmas 1a and 1b we have now gained at least some confidence about our methods for finding the right canonical form of $\overset{\vee}{f}$ to satisfy conjectures 1-3. Given this, let us take a moment to note a handful of further examples illustrating the power of these Taylor-series-to-the-left methods in calculating definite integrals (including some famous ones) which are otherwise difficult to evaluate.

Example 3(a) [Dirichlet's integral $\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x$]: The Taylor series for $f(x) = \frac{\sin x}{x}$ is $1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \ldots$ so we take $\overset{\vee}{f}(s) = \cos(\pi s)\cdot\cos(\pi\frac{s}{2})\cdot\frac{1}{(s+1)!}$. This has $\overset{\vee}{f}(-1) = 0$ and

$$\overset{\vee}{f}'(-1) = (-1)\cdot\lim_{\epsilon\to 0}\frac{\cos(\pi\frac{(-1+\epsilon)}{2})}{\epsilon} = -\frac{\pi}{2}$$

so we evaluate Dirichlet's integral as $\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x = \frac{\pi}{2}$. This is correct. Note that the evaluation is correct despite the integral not being absolutely convergent.

Example 3(b) [Fresnel's integral $\int_0^\infty \sin x^2\,\mathrm{d}x$]: The Taylor series for $f(x) = \sin x^2$ is $x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \ldots$ so using lemma 1a and shifting by 2 to the right, we take $\overset{\vee}{f}(s) = \frac{1}{8}\cdot\frac{\sin(2\pi(s-2))}{\sin(\frac{\pi(s-2)}{4})}\cdot\frac{1}{(\frac{s}{2})!}$. This has $\overset{\vee}{f}(-1) = 0$ and

$$\overset{\vee}{f}'(-1) = -\frac{\sqrt{2}}{8}\cdot\frac{1}{\sqrt{\pi}}\cdot\lim_{\epsilon\to 0}\frac{\sin(2\pi(-3+\epsilon))}{\epsilon} = -\sqrt{\frac{\pi}{8}}$$

so we evaluate Fresnel's integral as $\int_0^\infty \sin x^2\,\mathrm{d}x = \sqrt{\frac{\pi}{8}}$. This is correct and again this is despite the integral not being absolutely convergent. A similar calculation gives $\int_0^\infty \cos x^2\,\mathrm{d}x = \sqrt{\frac{\pi}{8}}$ also.

The generalised Fresnel integrals can be handled in identical fashion, giving $\int_0^\infty \sin x^n\,\mathrm{d}x = (\frac{1}{n})!\cdot\sin(\frac{\pi}{2n})$ and $\int_0^\infty \cos x^n\,\mathrm{d}x = (\frac{1}{n})!\cdot\cos(\frac{\pi}{2n})$.

Our final example in this group shows the application of Taylor-series-to-the-left methods in ways going beyond the use of lemmas 1a and 1b for handling re-scaling of well-known functions (on which we have principally relied so far). The working in this case involves formal manipulations, and is essentially combinatorial. It points towards the use of formal symbols and formal function elements in new ways in the final paper in this set.[3]

Example 3(c) [$\int_0^\infty \left(\mathbf{e}^{-\frac{1}{1+x^2}} - 1\right) \mathbf{d}x$]: Since $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots$ near 0, and since

$$\left(1 - x^2 + x^4 - x^6 + \ldots\right)^n = 1 - \binom{n}{n-1} x^2 + \binom{n+1}{n-1} x^4 - \binom{n+2}{n-1} x^6 - \ldots$$

so for x near 0 the Taylor series for $f(x) = \mathrm{e}^{-\frac{1}{1+x^2}}$ is given by

$$\begin{aligned} \mathrm{e}^{-\frac{1}{1+x^2}} &= \left\{ \begin{array}{c} 1 - \left(1 - x^2 + x^4 - x^6 + \ldots\right) + \frac{1}{2!}\left(1 - 2x^2 + 3x^4 - 4x^6 + \ldots\right) \\ -\frac{1}{3!}\left(1 - 3x^2 + 6x^4 - 10x^6 + \ldots\right) + \ldots \end{array} \right\} \\ &= \sum_{m=0}^{\infty} \overset{\vee}{f}(2m) x^{2m} \end{aligned}$$

where, formally,

$$\overset{\vee}{f}(2m) = (-1)^{m+1} \left\{ \begin{array}{c} \left\{ \ldots - \frac{1}{(-2)!}\binom{m-3}{m} + \frac{1}{(-1)!}\binom{m-2}{m} - \frac{1}{0!}\binom{m-1}{m} \right\} \\ + \left\{ \frac{1}{1!}\binom{m}{m} - \frac{1}{2!}\binom{m+1}{m} + \frac{1}{3!}\binom{m+2}{m} - \ldots \right\} \end{array} \right\}.$$

Here we have formally extended this expression to the left for reasons we cover in the next paper in this set (in order to match the correct asymptotic expansion for $f(x)$ as $x \to \infty$); and expressions involving binomial symbols with negative integer arguments are interpreted in the natural way so that, for example $\binom{-2}{-5} = \frac{(-2)!}{(-5)!\,3!} = \frac{(-2)(-3)(-4)}{3!} = -4$ and so forth (this can be justified as taking limiting values where the arguments approach these negative integer points in the appropriate way).

Bearing in mind lemma 1a, we thus set

$$\overset{\vee}{f}(s) = -\cos(\pi s) \cdot \cos(\pi \frac{s}{2}) \cdot \left\{ \begin{array}{c} \left\{ \ldots + \frac{1}{(-1)!}\binom{\frac{s}{2}-2}{\frac{s}{2}} - \frac{1}{0!}\binom{\frac{s}{2}-1}{\frac{s}{2}} \right\} \\ + \left\{ \frac{1}{1!}\binom{\frac{s}{2}}{\frac{s}{2}} - \frac{1}{2!}\binom{\frac{s}{2}+1}{\frac{s}{2}} + \frac{1}{3!}\binom{\frac{s}{2}+2}{\frac{s}{2}} - \ldots \right\} \end{array} \right\}. \quad (12)$$

[3] In that paper we will derive the formula for $\int_0^\infty \left(\mathrm{e}^{-\frac{1}{1+x^k}} - 1\right) \mathrm{d}x$ for arbitrary $k \in \mathbb{Z}_{\geq 1}$ in a matter of a few lines of working involving a formal symbol and purely using elementary algebra and combinatorics - despite the fact that, at least to the author, it is completely unclear how to obtain this formula using traditional integration methods for arbitrary k

This has $\overset{\vee}{f}(-1) = 0$ owing to the $cos(\pi\frac{s}{2})$ factor, and

$$\begin{aligned}\overset{\vee}{f}{}'(-1) &= \frac{\pi}{2}\cdot\left\{\begin{array}{c}\{\ldots + 0 - 0\} \\ \\ +\left\{\frac{1}{1!}(\frac{\frac{-1}{2}}{\frac{-1}{2}}) - \frac{1}{2!}(\frac{\frac{1}{2}}{\frac{-1}{2}}) + \frac{1}{3!}(\frac{\frac{3}{2}}{\frac{-1}{2}}) - \ldots\right\}\end{array}\right\} \\ &= \frac{\pi}{2}\cdot\left\{\begin{array}{c} 1 - \frac{1}{2!}\left(\frac{1}{2}\right) + \frac{1}{3!}\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\frac{1}{2!} \\ \\ -\frac{1}{4!}\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\frac{1}{3!} + \ldots\end{array}\right\} \simeq 1.258924257\,.\end{aligned}$$

We thus evaluate $\int_0^\infty \left(\mathrm{e}^{-\frac{1}{1+x^2}} - 1\right)\,\mathrm{d}x \simeq -1.258924257$ and this is readily confirmed as correct by numerical integration.

Comment: Note that we have considered here only $f(x) = \mathrm{e}^{-\frac{1}{1+x^2}}$, not the full integrand $g(x) := \mathrm{e}^{-\frac{1}{1+x^2}} - 1$, in calculating $\overset{\vee}{f}(s)$ and hence $\overset{\vee}{f}{}'(-1)$, and yet we obtained $\int_0^\infty \left(\mathrm{e}^{-\frac{1}{1+x^2}} - 1\right)\,\mathrm{d}x$ correctly.

One explanation for this is that the extra -1 term only affects, if anything, the $s = 0$ value of $\overset{\vee}{f}(s)$. As such its inclusion would leave $\overset{\vee}{f}(s)$ unaltered near $s = -1$ and thereby leave the value $\overset{\vee}{f}{}'(-1)$ unchanged.

There is, however, an alternative deeper explanation. The extra -1 has been included in the integrand solely to make it classically integrable - as $x \to \infty$, $\mathrm{e}^{-\frac{1}{1+x^2}} + C \sim (1+C) + O(\frac{1}{x^2})$ and so $\mathrm{e}^{-\frac{1}{1+x^2}} + C$ is classically integrable if and only if $C = -1$. The Taylor-series-to-the-left methods thus give us the correct value for the convergent integral when $C = -1$, but would give the same value even if a different constant C had been added.

Now, if we consider the partial integral $\int_0^X \left(\mathrm{e}^{-\frac{1}{1+x^2}} + C\right)\,\mathrm{d}x$, taking a different C only changes the result by $(1+C)X$ and $(1+C)X$ is a generalised Césaro eigenfunction with generalised Césaro limit 0. Thus, viewed within a generalised Césaro rather than a classical convergence framework, the integral-value *should* be independent of the choice of C.

This strongly suggests that our Taylor-series-to-the-left methods are in fact valid within the broader generalised Césaro framework, and that this is in fact the right framework in which to consider them. In other words, while they will certainly work where e.g. the integrand is classically integrable, they will continue to hold even where classical integrability fails but the integral converges in a generalised Césaro sense. We return to test this further shortly.

Our last two groups of examples in this section are motivated by the last two comments - following examples 2(a)-(c) and example 3(c) respectively.

The first of these noted that our Taylor-series-to-the-left methodology correctly calculated $\int_0^\infty f(x)\,\mathrm{d}x$ even when the Taylor series for f only converged for $|x| < 1$.

This represents the first hint of the potential for "local-to-global inference" from such methods. By this we mean that such integrals are quintessentially global - they depend critically on the behaviour of $f(x)$ on all of $[0, \infty)$, including the behaviour as $x \to \infty$. Yet Taylor-series-to-the-left methods appear to be built simply from an examination of the behaviour of $f(x)$ in an infinitesimal neighbourhood of 0 - from which we derive the values of $f^{(j)}(0)$ for all $j \geq 0$ and hence infer the form of $\overset{\vee}{f}(s)$ - and in this case only allow reconstruction of $f(x)$ for x on the small subset of the integration domain consisting of $[0, 1)$.

Our single example in group 4 takes this to an extreme by showing that Taylor-series-to-the-left methods continue to work even where the power series for $f(x)$ near 0 has radius of convergence $R = 0$ - so that it represents an asymptotic series governing the behaviour of $f(x)$ only in infinitesimally small neighbourhoods of 0, with the power series itself not being convergent for even a single value of $x > 0$!

To obtain this example, recall that in [IV], as a demonstration of using Césaro arrays based on an example from [2], we found an explicit function having the asymptotic power series $\sum_{j=0}^{\infty} B_{j+1} x^j = \sum_{j=0}^{\infty} (-1)^j (j+1) \zeta(-j) x^j = -\frac{1}{2} + \frac{1}{6}x - \frac{1}{30}x^3 + \ldots$ in a neighbourhood of $x = 0$. The coefficients in this series blow up rapidly as $j \to \infty$, and it can be easily shown that the series has radius of convergence $R = 0$, so that it is only an asymptotic series near 0 and does not converge for any $x \neq 0$.

Nevertheless we showed that $f(x)$ given by

$$f(x) = -\frac{1}{x} + \sum_{j=1}^{\infty} \frac{1}{(jx+1)^2}$$

is a perfectly well-defined function on all of $(0, \infty)$, which extends smoothly to 0 by its limiting value $-\frac{1}{2}$, and which has this power series as its asymptotic expansion near 0.

Unfortunately the $-\frac{1}{x}$ term here, while it cancels the $\frac{1}{x}$-divergence of $\sum_{j=1}^{\infty} \frac{1}{(jx+1)^2}$ as $x \to 0$, leaves a classically non-integrable divergence in $f(x)$ as $x \to \infty$, since $\sum_{j=1}^{\infty} \frac{1}{(jx+1)^2} \sim \zeta(2) \frac{1}{x^2}$ as $x \to \infty$. To avoid this and leave a classically integrable function on $[0, \infty)$ it would be better to subtract off instead a function which has the right $\frac{1}{x}$-divergence as $x \to 0$ but is integrable as $x \to \infty$.

Fortunately, we already know such a function, namely $2 \cdot \sum_{j=1}^{\infty} \mathrm{e}^{-\pi j^2 x^2}$ since in [IV] we showed that

$$\sum_{j=1}^{\infty} \mathrm{e}^{-\pi j^2 x^2} = \frac{1}{2}\frac{1}{x} - \frac{1}{2} + \mathcal{S}_0(x) \quad \text{and} \quad \sum_{j=1}^{\infty} \mathrm{e}^{-\pi j^2 x^2} = \mathcal{S}_\infty(x)$$

where $\mathcal{S}_0(x)$ and $\mathcal{S}_\infty(x)$ refer to the spaces of Schwartzian functions at 0 and ∞ respectively. Thus, finally, the example we consider is

Example 4(a) [$\int_0^\infty h(x)\,\mathrm{d}x$ where $h(x) := -2 \cdot \sum_{j=1}^{\infty} \mathrm{e}^{-\pi j^2 x^2} + \sum_{j=1}^{\infty} \frac{1}{(jx+1)^2}$]:

This function is smooth on $(0,\infty)$ and extends smoothly back to the value $\frac{1}{2}$ at $x=0$; but the Taylor coefficients of $x, x^2, x^3, \ldots$, remain $B_2, B_3, B_4, \ldots$ and so the radius of convergence of its Taylor series remains $R=0$ and it still does not converge for any $x \neq 0$. As for $\overset{\vee}{h}(s)$, we have

$$\overset{\vee}{h}(s) = \cos(\pi s) \cdot (s+1) \cdot \zeta(-s) - 2 \cdot \pi^{\frac{s}{2}} \cdot \cos(\pi s) \cdot \cos(\pi \frac{s}{2}) \cdot \frac{\zeta(-s)}{(\frac{s}{2})!}\,.$$

It follows, taking $s = -1+\epsilon$ and letting $\epsilon \to 0$, that we have

$$\begin{aligned}
\overset{\vee}{h}(-1) &= \lim_{\epsilon\to 0} \left\{ \begin{array}{c} (-1)\cdot\epsilon\cdot\left[-\frac{1}{\epsilon}+\gamma+\ldots\right] \\ \\ -2\cdot\pi^{-\frac{1}{2}}\cdot(-1)\cdot\left(\frac{\pi\epsilon}{2}\right)\cdot\left[-\frac{1}{\epsilon}+\gamma+\ldots\right]\cdot\frac{1}{\sqrt{\pi}} \end{array} \right\} \\
&= \lim_{\epsilon\to 0} \{(1+O(\epsilon)) + (-1+O(\epsilon))\} = 0
\end{aligned}$$

so $\overset{\vee}{h}(-1) = 0$ as it should (since h is integrable on $[0,\infty)$). And, working to $O(\epsilon^2)$ we have that

$$\begin{aligned}
\overset{\vee}{h}(-1+\epsilon) &= \left\{ \begin{array}{c} \cos(-\pi+\pi\epsilon)\cdot\epsilon\cdot\left[-\frac{1}{\epsilon}+\gamma+\ldots\right] \\ \\ -2\cdot\pi^{-\frac{1}{2}+\frac{\epsilon}{2}}\cdot\cos(-\pi+\pi\epsilon)\cdot\cos(-\frac{\pi}{2}+\frac{\pi\epsilon}{2})\cdot\frac{\left[-\frac{1}{\epsilon}+\gamma+\ldots\right]}{(-\frac{1}{2}+\frac{\epsilon}{2})!} \end{array} \right\} \\
&= \left\{ \begin{array}{c} (-1)\cdot(-1+\gamma\epsilon) \\ \\ -\frac{2}{\sqrt{\pi}}\cdot(1+\frac{1}{2}\ln(\pi)\epsilon)\cdot(-1)\cdot(\frac{\pi\epsilon}{2})\cdot\frac{\left[-\frac{1}{\epsilon}+\gamma+\ldots\right]}{(-\frac{1}{2}+\frac{\epsilon}{2})!}+O(\epsilon^2) \end{array} \right\}.
\end{aligned}$$

Now $(-\frac{1}{2}+\frac{\epsilon}{2})! = \Gamma(\frac{1}{2}+\frac{\epsilon}{2}) = \Gamma(\frac{1}{2})+\frac{1}{2}\Gamma'(\frac{1}{2})\epsilon+O(\epsilon^2)$, and from Gauss' di-gamma theorem we know that $\frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})} = -\gamma - 2\ln 2$. Thus, noting that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, we have $(-\frac{1}{2}+\frac{\epsilon}{2})! = \sqrt{\pi}\cdot(1-\frac{1}{2}(\gamma+2\ln 2)\epsilon)+O(\epsilon^2)$ and so $\frac{1}{(-\frac{1}{2}+\frac{\epsilon}{2})!} = \frac{1}{\sqrt{\pi}}\cdot(1+\frac{1}{2}(\gamma+2\ln 2)\epsilon))+O(\epsilon^2)$. It follows in the above that

$$\begin{aligned}
\overset{\vee}{h}(-1+\epsilon) &= \left\{ \begin{array}{c} \frac{2}{\pi}\cdot(1+\frac{1}{2}\ln(\pi)\epsilon)\cdot\left[-\frac{\pi}{2}+\frac{\pi\gamma}{2}\epsilon\right]\cdot(1+\frac{1}{2}(\gamma+2\ln 2)\epsilon)) \\ \\ +\,(1-\gamma\epsilon)+O(\epsilon^2) \end{array} \right\} \\
&= -\left(\frac{1}{2}\ln(4\pi)+\frac{1}{2}\gamma\right)\epsilon+O(\epsilon^2)
\end{aligned}$$

and so, finally,

$$\int_0^\infty h(x)\,\mathrm{d}x = -\overset{\vee}{h}{}'(-1) = -\lim_{\epsilon\to 0}\frac{\overset{\vee}{h}(-1+\epsilon)}{\epsilon} = \frac{1}{2}\ln(4\pi)+\frac{1}{2}\gamma \simeq 1.554119956\,.$$

This is again readily validated by numerical integration, confirming that Taylor-series-to-the-left methods continue to work here even when the radius of convergence of the series is zero. In passing, it is also interesting that they give such a clean closed form for the result. It would be much harder to obtain this via traditional methods.

Turning next to our second comment (regarding example 3(c)), our last set of examples show that Taylor-series-to-the-left methods continue to work for integrals even where they are only convergent in a generalised Césaro sense, not classically.

Example 5(a) [$\int_0^\infty \sin x\,\mathbf{d}x$]: The Taylor series for $f(x) = sinx$ is $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$ so, using lemma 1a and shifting 1 to the right, we take $\check{f}(s) = \frac{1}{4}\frac{\sin 2\pi(s-1)}{\sin\frac{\pi(s-1)}{2}} \cdot \frac{1}{s!}$. This has $\check{f}(-1) = 0$ and $\check{f}'(-1) = -1$, so conjecture 1 implies that we have

$$\int_0^\infty \sin x\,\mathrm{d}x = 1 \quad .$$

Of course $\int_0^\infty \sin x\,\mathrm{d}x$ is not classically convergent, but $\int_0^X \sin x\,\mathrm{d}x = [-\cos x]_0^X = 1 - \cos X$ and since $\cos X$ is strongly Césaro-convergent to 0 (since $P[\cos\tilde{X}](X) = \frac{1}{X}\int_0^X \cos x\,\mathrm{d}x = \frac{\sin X}{X} \to 0$ as $X \to \infty$), it follows that $\int_0^X \sin x\,\mathrm{d}x \overset{C}{\simeq} 1$ as $X \to \infty$. In other words, in a generalised Césaro sense, we do have that $\int_0^\infty \sin x\,\mathrm{d}x = 1$.

Thus, as in the discussion of example 3(c), conjecture 1 and our Taylor-series-to-the-left methods do seem to remain true for functions which are only Césaro-integrable; and generalised Césaro-convergence, rather than classical convergence, does seem to be the correct framework within which to view conjectures 1-3 and these methods.

In the same fashion, Taylor-series-to-the-left methods and conjecture 1 suggest that we should have $\int_0^\infty \cos x\,\mathrm{d}x = 0$ and, while this integral is not classically well-defined, it is Césaro-convergent and its generalised Césaro-value is 0, again supporting this perspective.

For our next, and final, example testing this generalised Césaro extension, we consider the always-interesting function $\ln(\Gamma(x+1))$. This is smooth on $[0,\infty)$ and by Stirling's theorem, it has asymptotic expansion

$$\ln(\Gamma(x+1)) = (x+\frac{1}{2})\ln x - x + \frac{1}{2}\ln 2\pi + \frac{1}{12}\frac{1}{x} + O(\frac{1}{x^3})$$

as $x \to \infty$.

It is thus clearly not classically integrable, but it is also not integrable in a generalised Césaro sense owing to the $\frac{1}{12}\frac{1}{x}$ term. In the partial-integral from 0 to X, the first terms lead to a linear combination of terms of the form $X^2\ln X$, $X\ln X$, X^2 and X, but these are all eigenfunctions or generalised eigenfunctions of P with eigenvalue either $\frac{1}{2}$ or $\frac{1}{3}$ and so have generalised Césaro limit 0; the $\frac{1}{12}\frac{1}{x}$ term, however, leads to a pure log-divergence $\frac{1}{12}\ln X$ as $X \to \infty$ and this is a generalised Césaro eigenfunction with eigenvalue 1 which cannot be attributed a generalised Césaro limit (see [I]-[III]).

To get a function which is not classically integrable but *is* Césaro-integrable, let us therefore take instead $f(x) := \ln(\Gamma(x+1)) - \frac{1}{12}\frac{1}{x+1}$, since this removes the log-divergence as $X \to \infty$ without disturbing smoothness and non-singularity near 0.

Example 5(b) [$\int_0^\infty f(x)\,\mathrm{d}x$ where $f(x) := \ln(\Gamma(x+1)) - \frac{1}{12}\frac{1}{x+1}$]: Since the Taylor series for $\ln(\Gamma(x+1))$ near $x=0$ is $-\gamma \cdot x + \sum_{m=2}^{\infty}(-1)^m \frac{\zeta(m)}{m} x^m$, and the corresponding Taylor series for $\frac{1}{1+x}$ is $1 - x + x^2 - x^3 + \ldots$, it is natural to try taking

$$\overset{\vee}{f}(s) = \cos(\pi s) \cdot \left(\frac{\zeta(s)}{s} - \frac{1}{12} \right)$$

except that there seems to be a problem with this prescription for $\overset{\vee}{f}(s)$ at $s=1$, since ζ has a pole there rather than the value γ.

It turns out, however, that this prescription for $\overset{\vee}{f}(s)$ is correct - and this apparent anomaly at $s=1$ in fact has interesting implications regarding the asymptotic behaviour of the function $\ln(\Gamma(x+1))$ as $x \to \infty$. We will begin to explain this in the next paper in the set; tighten the analysis into a formal result in the third paper; and finally analyse this specific case again in detail as an illustration of the power of local-to-global inference using Taylor-series-to-the-left methods in the fourth paper in the set.

For now, however, we take it as given that this definition of $\overset{\vee}{f}(s)$ (which works for all $s \in \mathbb{Z}_{>1}$) is correct and proceed accordingly.

We have at once that $\overset{\vee}{f}(-1) = 0$, and for $s = -1 + \epsilon$ we have

$$\begin{aligned} \overset{\vee}{f}(-1+\epsilon) &= (-1 + O(\epsilon^2)) \cdot \left\{ -\zeta(-1+\epsilon) \cdot \left[1 + \epsilon + O(\epsilon^2)\right] - \frac{1}{12} \right\} \\ &= (-1) \cdot \left\{ [-\zeta(-1) - \zeta'(-1)\epsilon] \cdot [1+\epsilon] - \frac{1}{12} \right\} + O(\epsilon^2) \\ &= -\left(\frac{1}{12} - \zeta'(-1) \right) \epsilon + O(\epsilon^2) \,. \end{aligned}$$

Now $\zeta'(-1) = \frac{1}{12} - \ln A$ where $A \approx 1.282427129$ is the Glaisher-Kinkelin constant. It follows that

$$\overset{\vee}{f}{}'(-1) = \lim_{\epsilon \to 0} \frac{\overset{\vee}{f}(-1+\epsilon)}{\epsilon} = -\ln A \approx -\ln(1.282427129)$$

and so conjecture 1 and our Taylor-series-to-the-left methodology imply that the generalised Césaro value of $\int_0^\infty f(x)\,\mathrm{d}x$ is

$$\int_0^\infty f(x)\,\mathrm{d}x = \ln A \approx \ln(1.282427129) \quad .$$

In light of the Stirling's theorem asymptotics for $\ln(\Gamma(x+1))$ given above, this can be re-stated as the claim that

$$\lim_{X\to\infty}\left\{\begin{array}{c}\int_0^X f(x)\,\mathrm{d}x-\frac{1}{2}X^2\ln X-\frac{1}{2}X\ln X\\ \\ +\frac{3}{4}X^2+(\frac{1}{2}-\frac{1}{2}\ln 2\pi)X\end{array}\right\}=\ln A\approx\ln(1.282427129)$$

where this is now a classical limit. This is again readily verified by numerical integration.[4]

Comment: Once more we see that Taylor-series-to-the-left methodology and its associated conjectures 1-3 seem to remain valid in the generalised Césaro convergence framework, and indeed to be best understood within this broader convergence framework.

Beyond the examples considered, such a generalisation from classical to generalised Césaro convergence appears plausible from a heuristic point of view. After all, arguing as we did in considering example 3(c), suppose conjecture 1 holds for all smooth, classically integrable functions; and suppose we take such a function, $f(x)$, and make it into a non-classically integrable function - but one which is still integrable in a generalised Césaro sense - by adding to its partial-integral, $F(X)$, a generalised Césaro eigenfunction of the form $c\cdot X^\rho$ for some constants c and $\rho>0$. That is, we replace $f(x)$ by $g(x)$ where $G(X)=F(X)+c\cdot X^\rho$.

Then $g(x):=f(x)+\rho x^{\rho-1}$ and since $\rho-1\neq-1$ the Taylor-coefficient functions of g and f are identical, at least in a neighbourhood of $s=-1$. Since, by conjecture 1 we have that $\int_0^\infty f(x)\,\mathrm{d}x=-\overset{\vee}{f}'(-1)$, and since $G(X):=\int_0^\infty g(x)\,\mathrm{d}x$ has generalised Césaro limit equal to the classical limit of $F(X)$ (since $X^\rho\overset{C}{\sim}0$), it follows that we also have $\int_0^\infty g(x)\,\mathrm{d}x=\underset{X\to\infty}{Clim}\int_0^X g(x)\,\mathrm{d}x=-\overset{\vee}{f}'(-1)=-\overset{\vee}{g}'(-1)$ and so conjecture 1 would continue to hold for $g(x)$ when understood within a generalised Césaro convergence framework.

In the same way, this heuristic argument could be extended to cover divergences consisting of generalised Césaro eigenfunctions, $c\cdot X^\rho(\ln X)^m$ ($m\in\mathbb{Z}_{>0}$), albeit that the form of $g(x)$ in terms of $f(x)$ in the argument is messier to state.

It thus appears that in order to extend our Taylor-series-to-the-left methodology and conjectures 1-3 from the classically convergent domain to that of generalised Césaro convergence, the only case which is not clear, at least morally, is the case where functions arise which are only Césaro-convergent under a pure power of P (such as the examples of $\sin x$ and $\cos x$ in example 5(a)). In such cases, the nature of the additional term in the partial-integral which is only convergent in a generalised Césaro sense, is not so easily inverted to take the integrand back into the domain of classical convergence.

[4] The convergence of the numerical calculation can be improved by including the anti-derivatives of extra decaying terms from Stirling's approximation in the expression on the left if desired

Nevertheless, as we saw in example 5(a), the Césaro extension of our Taylor-series-to-the-left methods and conjectures 1-3 does indeed hold in such cases, and we will prove this in the third paper in this set. There we will simultaneously show that the right convergence framework for understanding Mellin transforms is also the generalised Césaro, rather than the classical, convergence framework.

Let us now turn to consider such Mellin transforms in general, albeit reverting back for now to the warm embrace of classical convergence in order to avoid dividing our forces on too many fronts simultaneously.

4 The Taylor coefficient function and Mellin transforms

One of the most important areas where integrals on $[0,\infty)$ occur is the Mellin transform. For a suitably well-behaved, smooth function $f(x)$ its Mellin transform, $\mathcal{M}[f](s)$, is a function of complex variable s defined by

$$\mathcal{M}[f](s) := \int_0^\infty x^{s-1} f(x)\,\mathrm{d}x\,. \tag{13}$$

Since by conjecture 1 we have that $\int_0^\infty f(x)\,\mathrm{d}x = -\overset{\vee}{f}'(-1)$, and since the Taylor series for $g(x) := x^{s-1}f(x)$ might be expected to be just the Taylor series for $f(x)$ shifted to the right by $s-1$, we might expect to have simply $\mathcal{M}[f](s) = \overset{\vee}{f}'(-s)$ in general. Indeed when $s = m \in \mathbb{Z}$ this is the case, as we claimed earlier in section 2.

For $s \notin \mathbb{Z}$, however, it is not correct. The reason is that when $s \notin \mathbb{Z}$, $g(x) = x^{s-1}f(x)$ is no longer either smooth near $x = 0$, nor contains only poles there. Rather, $x = 0$ is now a branch point and $g(x)$ has a branch cut originating there. This makes the notion of a Taylor or Laurent series for $g(x)$ around 0 problematic. Since the reasoning of section 2 only applied where such a power series exists, we cannot simply apply such translation-arguments when $s \notin \mathbb{Z}$.

We can, however, get around this, and bootstrap from our result in conjecture 1 to a general formula relating the Taylor-coefficient function of a function, f, to its Mellin transform at arbitrary s.

The key is to start by restricting to $s \in \mathbb{Q}$ rational and then to use a substitution which converts this case back to one involving only integer powers to which conjecture 1 can be applied. Let us illustrate first by deriving the well-known Mellin transform of e^{-x} by these means.

Example 6 [The Mellin transform of $f(x) = \mathrm{e}^{-x}$]: For $s = \frac{p}{q}$ in lowest terms, we have

$$\mathcal{M}[\mathrm{e}^{-x}](\frac{p}{q}) = \int_0^\infty x^{\frac{p}{q}-1}\mathrm{e}^{-x}\,\mathrm{d}x \quad .$$

Letting $u = x^{\frac{1}{q}}$, we get $\mathrm{d}x = qu^{q-1}\mathrm{d}u$ and therefore

$$\mathcal{M}[f](\frac{p}{q}) = q \cdot \int_0^\infty u^{p-1} \cdot \mathrm{e}^{-u^q}\,\mathrm{d}u \quad .$$

Now $h(u) := u^{p-1} \cdot \mathrm{e}^{-u^q} = u^{p-1} \cdot \{1 - u^q + \frac{1}{2!}u^{2q} - \frac{1}{3!}u^{3q} + \ldots\}$ has only integer powers and so, using lemma 1a and shifting $p-1$ to the right, we take

$$\overset{\vee}{h}(s) = \frac{1}{2q} \cdot \frac{\sin(2\pi(s+1-p))}{\sin(\pi\frac{(s+1-p)}{q})} \cdot \frac{1}{(\frac{s+1-p}{q})!} \quad .$$

This has $\overset{\vee}{h}(-1) = 0$ and

$$\overset{\vee}{h}(-1+\epsilon) = \frac{1}{2q} \cdot \frac{2\pi\epsilon}{\sin(-\pi\frac{p}{q})} \cdot \frac{1}{(\frac{-p}{q})!} + O(\epsilon^2) \quad .$$

Thus

$$\overset{\vee}{h}'(-1) = \lim_{\epsilon\to 0} \frac{\overset{\vee}{h}(-1+\epsilon)}{\epsilon} = -\frac{1}{q} \cdot \frac{\pi}{\sin(\pi\frac{p}{q})} \cdot \frac{1}{(\frac{-p}{q})!} = -\frac{1}{q} \cdot \Gamma(\frac{p}{q})$$

on using the functional equation of the Gamma function as before. Hence we have that

$$\mathcal{M}[f](\frac{p}{q}) = q \cdot \int_0^\infty h(u)\,\mathrm{d}u = \Gamma(\frac{p}{q})$$

for any $\frac{p}{q} \in \mathbb{Q}$. Since for any $s \in \mathbb{R}$ we can approximate s arbitrarily closely by rational values, and since the result just obtained depends only on $\frac{p}{q}$, not on p or q independently, it follows by taking $\frac{p}{q} \to s$ that for arbitrary $s \in \mathbb{R}$ (and hence also arbitrary $s \in \mathbb{C}$ by analytic continuation) we have likewise that

$$\mathcal{M}[f](s) = \Gamma(s)\,.$$

We have thus derived the well-known result giving $\Gamma(s)$ as the Mellin transform of e^{-x}. Let us apply this same bootstrapping approach to an arbitrary function $f(x)$.

The general case [The Mellin transform of $f(x)$]: For $s = \frac{p}{q}$ in lowest terms, the same substitution of $u = x^{\frac{1}{q}}$ gives us

$$\mathcal{M}[f](\frac{p}{q}) = q \cdot \int_0^\infty u^{p-1} \cdot f(u^q)\,\mathrm{d}u \quad .$$

Now $h(u) := u^{p-1} \cdot f(u^q) = u^{p-1} \cdot \left\{\overset{\vee}{f}(0) + \overset{\vee}{f}(1)u^q + \overset{\vee}{f}(2)u^{2q} + \overset{\vee}{f}(3)u^{3q} + \ldots\right\}$ has only integer powers but is not necessarily alternating in character. Thus we need to invoke lemma 1b rather than lemma 1a while still shifting $p-1$ to the right, in constructing $\check{h}(s)$. We get

$$\check{h}(s) = \frac{1}{q} \cdot \frac{\sin(2\pi(s+1-p))}{\sin(2\pi\frac{(s+1-p)}{q})} \cdot \check{f}(\frac{s+1-p}{q}) \quad .$$

Thus, for $q > 1$ we have that $\check{h}(-1) = 0$ and

$$\check{h}(-1+\epsilon) = \frac{1}{q} \cdot \frac{2\pi\epsilon}{\sin(-2\pi\frac{p}{q})} \cdot \check{f}(\frac{-p}{q}) + O(\epsilon^2)$$

so that

$$\check{h}'(-1) = \lim_{\epsilon\to 0} \frac{\check{h}(-1+\epsilon)}{\epsilon} = -\frac{1}{q} \cdot \check{f}(\frac{-p}{q}) \cdot \frac{2\pi}{\sin(2\pi\frac{p}{q})} \quad .$$

It follows that

$$\mathcal{M}[f](\frac{p}{q}) = -q \cdot \check{h}'(-1) = \check{f}(\frac{-p}{q}) \cdot \frac{2\pi}{\sin(2\pi\frac{p}{q})}$$

and since this again depends smoothly only on $\frac{p}{q} \in \mathbb{Q}$ (not on p or q independently), we can take $\frac{p}{q} \to s$ and readily show that in general

$$\mathcal{M}[f](s) = \frac{2\pi}{\sin(2\pi s)} \cdot \check{f}(-s) \tag{14}$$

for arbitrary $s \in \mathbb{R}$ (and hence arbitrary $s \in \mathbb{C}$ by analytic continuation).

Alternatively, this relationship can be re-expressed to give $\check{f}$ in terms of the Mellin transform, in what is now the canonical form for $\check{f}$ we have been seeking:

Canonical form for $\check{f}$:

$$\check{f}(s) = -\mathcal{M}[f](-s) \cdot \frac{\sin(2\pi s)}{2\pi} \quad . \tag{15}$$

4.1 Implications

This canonical form in equation 15 changes the standing of all the work preceding it in this paper. Previously, the extension from a well-defined Taylor series to a Taylor-coefficient function of a continuous variable was heuristic - and we had been primarily engaged in developing rules (e.g. lemmas 1a and 1b) which would allow us to obtain this correct form making conjectures 1-3 true.

Now equation 15 gives us the general formula for this canonical form - for any function f whose Mellin transform exists - and under this, conjecture 1 is readily validated as a theorem.

For if $\mathcal{M}[f](s)$ is well-defined and differentiable at $s = 1$ (which is essentially equivalent to our earlier integrability condition that $\overset{\vee}{f}(-1) = 0$), then differentiating equation 15 and setting $s = -1$ gives that

$$\overset{\vee}{f}'(-1) = -\left\{ \frac{d}{ds}\mathcal{M}[f](-s) \mid_{s=-1} \right\} \cdot 0 - \mathcal{M}[f](1) \cdot 1 = -\int_0^\infty f(x)\,\mathrm{d}x$$

and this proves conjecture 1. Conjectures 2 and 3 then follow as before; and the fact that, where applicable, lemmas 1a and 1b lead to the correct canonical form of $\overset{\vee}{f}(s)$ can be inferred from the way we just used lemma 1b to derive equation 15 above.

In short, we have now moved the notion of the Taylor-coefficient function from the realm of morally-justified intuition, to the realm of well-defined quantity and have at the same time moved conjectures 1-3 into the category of established results.

And at the same time, in ways that we will explore in greater depth in later papers in this set, we have seen via various examples that the two closely-connected concepts of Taylor coefficient functions and Mellin transforms should likely both be extended from the domain of classical convergence to the framework of generalised Césaro convergence; extended to encompass power series that may only be asymptotic and nowhere-convergent; and analysed in ways that may reveal the potential for local-to-global inference.

Comment: One last comment about aesthetics. We have remarked how conjecture 1 and Taylor-series-to-the-left methods facilitate calculation of integrals on $[0, \infty)$ without using anything resembling traditional integration techniques. It is therefore a little unsatisfying that the derivation of the Mellin transform in this section has relied on using a traditional substitution to first go from rational to integer exponents. In fact, this could have been avoided. As long as we have a "base unit" (in our case $\frac{1}{q}$) which evenly divides both 1 and all of the powers of x in our integrand, then lemmas 1a and 1b can be easily adapted to give sequences with ± 1 every n lengths of our base unit and zeros in between (see Appendix); and thereby to get the Taylor-coefficient function of $x^{\frac{p}{q}-1}f(x)$ directly. We omit details here, however. Of course, nothing about Taylor-series-to-the-left methods is incompatible with traditional techniques[5], so aesthetic purity for its own sake is generally inadvisable in any case.

5 Definite integrals more generally

Not all definite integrals have domain $[0, \infty)$. Other cases, however - such as $\int_a^\infty f(x)\,\mathrm{d}x$ or $\int_a^b f(x)\,\mathrm{d}x$ or $\int_{-\infty}^a f(x)\,\mathrm{d}x$ - can still be handled by combining Taylor-series-to-the-left methods with traditional substitutions.

[5]providing we take care - see next section

For example, under trivial substitutions $\int_a^\infty f(x)\,\mathrm{d}x$ becomes $\int_0^\infty f(u+a)\,\mathrm{d}u$; likewise $\int_a^b f(x)\,\mathrm{d}x$ becomes $\int_0^\infty f(u+a)\,\mathrm{d}u - \int_0^\infty f(u+b)\,\mathrm{d}u$; and $\int_{-\infty}^a f(x)\,\mathrm{d}x$ becomes $\int_0^\infty f(-u+a)\,\mathrm{d}u$. And similarly, integrals of the form $\int_0^1 f(\ln x)\,\mathrm{d}x$ transform, under $u = -\ln x$, into $\int_0^\infty f(-u)\mathrm{e}^{-u}\,\mathrm{d}u$; and so on.

In principle, then, Taylor-series-to-the-left methods can be applied, via substitution, to many definite integrals.

In practice, however, this may be difficult and certain issues must be considered:

Issues: (i) Such substitutions may introduce products, quotients or composite functions, and deriving the canonical Taylor-coefficient function of such combinations from those of the building blocks can be challenging.

For example, even for such a clean example of the form just mentioned as $\int_0^1 \frac{\sin(\ln x)}{\ln x}\,\mathrm{d}x$, this becomes

$$\int_0^\infty g(u)\,\mathrm{d}u \quad \text{where} \quad g(u) = \frac{\sin(u)}{u} \cdot \mathrm{e}^{-u}$$

and evaluating this requires us to calculate the Taylor-coefficient function of a product.

In this instance it can be done. We have $\frac{\sin(u)}{u} = 1 - \frac{u^2}{3!} + \frac{u^4}{5!} - \ldots$ and $\mathrm{e}^{-u} = 1 - u + \frac{u^2}{2!} - \frac{u^3}{3!} + \ldots$, so it is readily deduced that when $m \in \mathbb{Z}_{\geq 0}$, $\overset{\vee}{g}(m)$ is given by the finite sum

$$\overset{\vee}{g}(m) = (-1)^m \cdot \left\{ \frac{1}{m!1!} - \frac{1}{(m-2)!3!} + \frac{1}{(m-4)!5!} - \ldots \right\} \quad .$$

Taking $\overset{\vee}{g}(s)$ therefore as

$$\overset{\vee}{g}(s) = \cos(\pi s) \cdot \left\{ \frac{1}{s!1!} - \frac{1}{(s-2)!3!} + \frac{1}{(s-4)!5!} - \ldots \right\}$$

we have $\overset{\vee}{g}(-1) = 0$ and $\overset{\vee}{g}(-1+\epsilon) = (-1) \cdot \left\{ \epsilon - \frac{\epsilon}{3} + \frac{\epsilon}{5} - \ldots \right\}$, so that $\overset{\vee}{g}'(-1) = (-1) \cdot \left\{ 1 - \frac{1}{3} + \frac{1}{5} - \ldots \right\} = -\frac{\pi}{4}$ and thus finally

$$\int_0^1 \frac{\sin(\ln x)}{\ln x}\,\mathrm{d}x = \frac{\pi}{4} \quad .$$

For more complicated integrals, however - and especially those involving composite functions - the task of isolating the relevant canonical Taylor-coefficient function can be considerably more delicate and difficult. Thus Taylor-series-to-the-left methods do not represent a panacaea. We consider elements of a "calculus" for how Taylor-coefficient functions behave under basic operations (products, quotients, composition etc) in later papers in this series, to augment the results of lemmas 1a and 1b derived earlier regarding their behaviour under re-scaling.

(ii) Secondly, if we apply Taylor-series-to-the-left methods within the generalised Césaro-convergence framework, we must take care even with elementary translation-substitutions since generalised Césaro limits are not translation-invariant.[6]

For example, consider the case of $\int_0^1 \ln(\Gamma(x))\,\mathrm{d}x$, which is known to have value $\frac{1}{2}\ln(2\pi)$. This can be deduced from our earlier result for example 5(b) where we used Taylor-series-to-the-left methods to show that $\int_0^\infty \ln(\Gamma(x+1))\,\mathrm{d}x = -\ln A \approx -1.282$; but only if we take care that this is a generalised Césaro evaluation of an otherwise classically-divergent integral, so that $\int_0^\infty \ln(\Gamma(x+1))\,\mathrm{d}x$ is to be interpreted as

$$\underset{X\to\infty}{\mathit{Clim}} \left\{ \int_0^X \ln(\Gamma(x+1))\,\mathrm{d}x \right\} = \underset{X\to\infty}{\mathit{Clim}} \left\{ \begin{array}{c} -\ln A + \frac{1}{2}X^2\ln X + \frac{1}{2}X\ln X \\ -\frac{3}{4}X^2 - (\frac{1}{2} - \frac{1}{2}\ln(2\pi))X \end{array} \right\}.$$

It follows, on noting that $\ln(\Gamma(x)) = \ln(\Gamma(x+1)) - \ln x$ and performing a translation-substitution in the partial integral for the second integral, that

$$\begin{aligned} \int_0^1 \ln(\Gamma(x))\,\mathrm{d}x &= \int_0^\infty \ln(\Gamma(x))\,\mathrm{d}x - \int_1^\infty \ln(\Gamma(x))\,\mathrm{d}x \\ &= \underset{X\to\infty}{\mathit{Clim}} \left\{ \begin{array}{c} \int_0^X \{\ln(\Gamma(x+1)) - \ln x\}\,\mathrm{d}x \\ -\int_0^{X-1} \ln(\Gamma(x+1))\,\mathrm{d}x \end{array} \right\} \end{aligned}$$

and this therefore becomes

$$\begin{aligned} &\underset{X\to\infty}{\mathit{Clim}} \left\{ \begin{array}{c} \left[\begin{array}{c} -\ln A + \frac{1}{2}X^2\ln X + \frac{1}{2}X\ln X \\ -\frac{3}{4}X^2 - (\frac{1}{2} - \frac{1}{2}\ln(2\pi))X \end{array} \right] - [X\ln X - X] \\ - \left[\begin{array}{c} -\ln A + \frac{1}{2}(X-1)^2\ln(X-1) + \frac{1}{2}(X-1)\ln(X-1) \\ -\frac{3}{4}(X-1)^2 - (\frac{1}{2} - \frac{1}{2}\ln(2\pi))(X-1) \end{array} \right] \end{array} \right\} \\ = \ &\frac{1}{2}\ln(2\pi) \end{aligned}$$

where this last limit has now in fact become a classical limit, as it should.

Comment: The primary purpose of this example was to show the care that needs to be taken, in the generalised Césaro context, to apply substitutions in partial-integrals before passing to $\underset{X\to\infty}{\mathit{Clim}}$. The result is also interesting, however, in evaluating the integral of $\ln(\Gamma(x))$ on $[0,1]$ not only via a mixture of

[6]They are dilation- and scaling-invariant, but not translation-invariant

Taylor-series-to-the-left and generalised Césaro methods (with minimal reliance on traditional integration techniques), but also as the value of the constant term in Stirling's asymptotic expansion for $\ln(\Gamma(x))$ as $x \to \infty$ (which is as far from $[0,1]$ as you can get!). This is another illustration of the local-to-global connectedness realised by this combination of methods.

6 Final remarks - Taylor series to the left and Ramanujan's master theorem

As pointed out by Professor Sir Purun Dass[7] (pers. comm.), there is a close connection between, on the one hand, the notion of a Taylor-coefficient function developed in this paper and its application in conjecture 1, and, on the other, Ramanujan's master theorem (RMT) concerning the evaluation of Mellin transforms. Roughly speaking, the latter states:

Ramanujan's Master Theorem [RMT]: *If $f(x)$ can be expressed as the series $f(x) = \sum_{n=0}^{\infty} \phi(n) \frac{(-x)^n}{n!}$ for some well-behaved function $\phi(s)$ of a continuous variable s, then we have*

$$\mathcal{M}[f](s) = \int_0^{\infty} x^{s-1} f(x)\,\mathrm{d}x = \Gamma(s)\phi(-s) \tag{16}$$

for $Re(s)$ in that region where $x^{s-1}f(x)$ is classically integrable on $[0,\infty)$.

Clearly $\phi(s)$ plays a role very similar to the Taylor-coefficient function $\overset{\vee}{f}(s)$ and setting $s = 1$ in equation 16 recovers $\int_0^{\infty} f(x)\,\mathrm{d}x$ as $\phi(-1)$, so that we get the same focus on behaviour of ϕ at or near -1 in calculating this integral as we saw for $\overset{\vee}{f}$ in conjecture 1. Indeed, taking $\phi(s) = \frac{1}{\cos(\pi s)} \cdot s! \cdot \overset{\vee}{f}(s)$ makes this equivalence explicit; and the reader can see how the two approaches play out analogously under this equivalence.

Of course, the application of the RMT in the sort of examples we have considered often plays out a little differently from the approach we have taken. For example, for the general Fresnel integral $\int_0^{\infty} \sin(x^n)\,\mathrm{d}x$ discussed in example 3(b), the RMT is generally applied by setting $u = x^n$ to transform the integral into $\frac{1}{n}\int_0^{\infty} u^{\frac{1}{n}-1} \sin(u)\,\mathrm{d}u$ and then applying the RMT directly to $g(u) = \sin(u)$ with $s = \frac{1}{n}$. By contrast, our approach has been effectively to set $s = 1$ in the Mellin transform and tackle the calculation of $\overset{\vee}{f}(s)$ directly for $f(x) = \sin(x^n)$ using lemmas 1a and 1b.

[7]KCIE, DCL, Ph.D., etc, once Prime Minister of the progressive and enlightened State of Mohiniwala, and honorary or corresponding member of more learned and scientific societies than will ever do any good in this world or the next.

Nonetheless, at core, the two approaches are closely related and so the question arises of whether there is any value in the re-casting in terms of Taylor-coefficient functions which we have begun in this paper?

We believe strongly that there is value; and that the work in this paper and the others in this set is neither redundant nor pointless. Rather, it is worth undertaking both for its own sake, and also because it contains several aspects which take it into new territory and yield new results beyond purely the content of RMT (or conjectures 1-3). Specifically, there are four main justifications for this belief:

Justifications: a) First, the discussion in section 2 explains morally *why* the results of RMT and conjectures 1-3 hold. Presentation of the RMT often leaves it feeling mysterious and quasi-magical, so re-casting it in a natural framework which motivates it heuristically is valuable in and of itself.

A consequence of this is also that its re-casting via Taylor-coefficient functions and conjecture 1 then naturally connects to the extensions given in conjectures 2-3, and to the sort of extensions discussed in section 5, whereas such connections are somewhat hidden in the RMT and its usual development relating solely to Mellin transforms.

(b) Beyond heuristic coherence, the re-casting in terms of Taylor-coefficient functions and Taylor-series-to-the-left methods via lemmas 1a and 1b also facilitates new techniques for evaluating integrals.

For instance, in example 3(c) and in the calculation of $\int_0^1 \frac{\sin(\ln x)}{\ln x}\,\mathrm{d}x$ performed in the last section, we have seen how we could calculate Taylor-coefficient functions and hence definite integrals for integrand functions which are, respectively, a complicated composite function ($\mathrm{e}^{-\frac{1}{1+x^2}}$) and a non-trivial product ($\frac{\sin(u)}{u} \cdot \mathrm{e}^{-u}$). It is much harder (and much less natural) to see how to perform these calculations starting with the RMT.

In short, the focus here on Taylor-coefficient functions, $\overset{\vee}{f}(s)$, is much better-adapted than the RMT's focus on the quantity $\phi(s)$ for developing a "calculus" of how these quantities behave under combination via products, quotients, composition etc. Moreover, that calculus (which we will develop further in later papers) lends itself naturally, as we saw in the examples just listed, to combinatorial and formal methods for evaluation of integrals which we believe are essentially new. We will also develop these further - by connecting to the use of formal symbols and formal function elements - in later papers in this set.

(c) Thirdly, we have seen that our Taylor-series-to-the-left methods and conjectures 1-3 in this paper all appear to extend from the domain of classical convergence to the broader generalised Césaro convergence framework. This is in fact true (as we will show in the third paper in this set) and represents a significant extension of the RMT as traditionally stated.

It also implies at once that the true native habitat of the Mellin transform is likewise the lush, verdant pastures of the generalised Césaro convergence framework rather than the rocky ground and meagre grains of the fields of classical convergence. This we will also demonstrate in the third paper in this set. It leads not only to a significant extension of the domain of applicability of Mellin transforms, but also to a much cleaner treatment of them - one in which many of the technicalities and domain-restrictions of their classical treatment disappear.

(d) Finally, in the next paper in this set we will see that $\overset{\vee}{f}(s)$ has very nice properties that connect the behaviour of $f(x)$ near 0 and ∞ *simultaneously.*

This, in addition to the reasons given above in (b), suggests that $\overset{\vee}{f}(s)$ is in fact the "right" quantity to study rather than $\phi(s)$, which does not so readily connect with such aspects of the underlying behaviour of $f(x)$ and so does not seem to be much studied in its own right outside immediate applications of the RMT.

Indeed, while there are theorems built from generalisations of the RMT which, for example, connect the behaviour of a function of a complex variable with simple poles of known residues at the negative integer points to the power series expansion around $x = 0$ of the function essentially representing its inverse Mellin transform (see e.g. the final theorem in section 10.10 of [2] for this case), such theorems do not fall into a simple framework which renders them natural and intuitive; they are generally beset by significant technicalities regarding growth restrictions; and they focus only on behaviour either near $x = 0$ (as in the above case) or near $x = \infty$ as separate theorems, rather than considering behaviour at these points *simultaneously.*

By contrast, in the next three papers in this series, by recasting in terms of $\overset{\vee}{f}(s)$ and Taylor-series-to-the-left methodology we will derive analogous results which are natural and intuitive; are much less technically-restrictive; and which simultaneously connect behaviour of $f(x)$ near $x = 0$ and $x = \infty$ in an essential way.

7 Appendix

We noted in a comment at the end of section 4 that lemmas 1a and 1b could be generalised slightly to cover the case of a "base unit" length other than 1, as long as that base length still evenly divides both 1 and all the powers of x in the integrand-function being considered. Specifically, this generalisation could be stated as follows:

Lemma 1c: *Suppose we have a base unit $\frac{1}{q}$, $q \in \mathbb{Z}_{>0}$. Then for any $n \in \mathbb{Z}_{\geq 1}$ the function $\frac{1}{n} \cdot \frac{\sin(2\pi sq)}{\sin(\frac{2\pi sq}{n})}$ is smooth and has the value 1 at all the points $s = j\frac{n}{q}$ ($j \in \mathbb{Z}$); and has value 0 at all points of the form $s = k\frac{n}{q}$, $k \neq 0 \,(mod\, n)$. Thus this function has value 1 at all points occurring periodically every n multiples of the base-unit length either side of 0, and value 0 at all other points on the lattice of base-unit length $\frac{1}{q}$.*

This generalises lemma 1b. The corresponding generalisation of lemma 1a uses the function $\frac{1}{2n} \cdot \frac{\sin(2\pi sq)}{\sin(\frac{\pi sq}{n})}$. In either case the behaviour can be shifted to the right by $\frac{p}{q}$ by replacing s with $(s - \frac{p}{q})$ in these functions.

8 Acknowledgements

We thank Professors Peachy Carnahan, Daniel Dravot and Sir Purun Dass for many helpful suggestions (pers. comm.), and Professor T. Abby for his help in preparing this paper.

References

[I] R. Stone, *Introduction to generalised Césaro convergence I*, 2026

[II] R. Stone, *Introduction to generalised Césaro convergence II*, 2026

[III] R. Stone, *Introduction to generalised Césaro convergence III*, 2026

[IV] R. Stone, *Césaro Arrays I*, 2026

[V] R. Stone, *Césaro Arrays II*, 2026

[VI] R. Stone, *Césaro Arrays III*, 2026

[VII] R. Stone, *Root Identities I*, 2026

[VIII] R. Stone, *Root Identities II: Root identities for ζ - Part A*, 2026

[IX] R. Stone, *Root Identities II: Root identities for ζ - Part B*, 2026

[X] R. Stone, *Root Identities II: Root identities for ζ - Part C*, 2026

[1] I.S. Gradshteyn and I.M. Ryzhik, *Tables of Integrals, Series, and Products, Seventh Edition, Edited by Alan Jeffrey and Daniel Zwillinger*, Academic Press, 2007

[2] H.M. Edwards, *Riemann's Zeta Function*, Academic Press, 1974