Taylor Series to the Left II - behaviour approaching infinity

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Abstract

We show experimentally that the Taylor-coefficient function, $\overset{\vee}{f}(s)$, of a function $f(x)$ appears to encode information not just about the behaviour of $f(x)$ near $x = 0$, but also about its behaviour as $x \to \infty$. We explore this in a variety of examples and show that it can be used to infer asymptotic behaviour as $x \to \infty$ from behaviour near 0. We discuss preliminary aspects of this phenomenon of "local-to-global" inference using Taylor-series-to-the-left; and we use this discussion to craft a precise conjecture. Finally, we consider examples showing how behaviour of the Taylor-coefficient function in this conjecture relates, on the one hand, to generalised Césaro considerations and, on the other, to behaviour of the Mellin transform $\mathcal{M}[f](s)$.

1 Introduction

This is the second in a set of papers on the notion of Taylor-coefficient functions and Taylor-series-to-the-left methods. After introducing these concepts and showing how they relate to integration and the Mellin transform of a function $f(x)$ in [XI], this paper is very short. We give examples showing that these ideas also connect directly to the asymptotic behaviour of $f(x)$ as $x \to \infty$. From these we develop a general conjecture and we explore it in further examples showing its utility for non-trivial local-to-global inference and its underlying connection to the generalised Césaro framework. These ideas will all be clarified and proven in the following paper in the set.

In section 2, we consider three very simple examples and from these we develop and refine a conjecture linking Taylor-coefficient functions to the power series behaviour of $f(x)$ near $x = 0$ and near $x = \infty$ *simultaneously.* We discuss the implications of this conjecture for local-to-global inference of asymptotic behaviour near ∞ from behaviour near 0.

In section 3 we demonstrate this potential for local-to-global inference in a non-trivial example. We combine Taylor-series-to-the-left methods and this conjecture with earlier work on Césaro arrays to show how we identified in [IV] an explicit function having the asymptotic power series (with radius of convergence 0) given by $\sum_{j=0}^{\infty} B_{j+1}x^j$. This had been left unexplained in [IV] and only verified there after-the-fact.

In section 4, we consider one further elementary example. This suggests how we should interpret the case where the Taylor-coefficient function develops a pole singularity. It also further strengthens the connection with generalised Césaro convergence discussed and illustrated in [XI], and with it the associated connection with Mellin transform behaviour.

1.1 Final notes

We thank Prof Danny Deever of the Kafiristan Institute of Mathematics (KIM) for many helpful comments and suggestions (pers. comm.), and in particular for his recommendation that we introduce these ideas in this short exploratory paper, rather than leave them hanging until accompanied by formal proof.

2 Examples and a conjecture

Let us analyse some simple examples.

Example 1a [$f(x) = \frac{1}{1+x}$]: The function $f(x) = \frac{1}{1+x}$ is smooth on $[0, \infty)$[1]. Its Taylor series near 0 is $f(x) = 1 - x + x^2 - x^3 + \ldots$ with radius of convergence $R = 1$ and, per [XI], its canonical Taylor-coefficient function is $\overset{\vee}{f}(s) = \cos(\pi s)$. Therefore its Taylor series to the left is given by

$$f_{left}(x) = \sum_{m<0} \overset{\vee}{f}(m)x^m = -\frac{1}{x} + \frac{1}{x^2} - \frac{1}{x^3} + \frac{1}{x^4} - \ldots \quad . \tag{1}$$

On the other hand, for x large we have $\frac{1}{1+x} = \frac{1}{x\left(1+\frac{1}{x}\right)} = \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} - \frac{1}{x^4} + \ldots$, so we see that in this case *the Taylor series to the left for $f(x)$ near 0 gives the negative of the power series expansion for $f(x)$ as $x \to \infty$.*

Example 1b [$f(x) = \tan^{-1}(x)$]: Next consider the case of $f(x) = \tan^{-1}(x)$, also smooth on $[0, \infty)$. Its Taylor series near 0 is $f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$ with radius of convergence $R = 1$ and, again per [XI], its canonical Taylor-coefficient function is $\overset{\vee}{f}(s) = \cos(\pi(s-1)) \cdot \cos(\pi\frac{(s-1)}{2}) \cdot \frac{1}{s}$. For $s = \epsilon$ near 0 this has $\lim\limits_{\epsilon\to 0}\overset{\vee}{f}(\epsilon) = (-1) \cdot \lim\limits_{\epsilon\to 0}\frac{\frac{\pi\epsilon}{2}}{\epsilon} = -\frac{\pi}{2}$ and thus the Taylor series to the left for $f(x)$ is given in this instance by

$$f_{left}(x) = \sum_{m\leq 0} \overset{\vee}{f}(m)x^m = -\frac{\pi}{2} + \frac{1}{x} - \frac{1}{3}\frac{1}{x^3} + \frac{1}{5}\frac{1}{x^5} - \ldots \quad . \tag{2}$$

Now for x large we have $\tan^{-1}(x) = \frac{\pi}{2} - \tan^{-1}(\frac{1}{x}) = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3}\frac{1}{x^3} - \frac{1}{5}\frac{1}{x^5} + \ldots$ and so once again we have that *the Taylor series to the left for $f(x)$ near 0 gives*

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[1] This notation implies that f is smooth in an open neighbourhood of 0.

the negative of the power series expansion for $f(x)$ *as* $x \to \infty$.

Based on examples 1a and 1b we might boldly conjecture that, given the canonical form[2] of the Taylor-coefficient function, $\overset{\vee}{f}(s)$, of a smooth function $f(x)$ on $[0, \infty)$, its power series expansion near 0 might be given by the Taylor series to the right, $\sum_{m \geq m_0} \overset{\vee}{f}(m) x^m$, for some index m_0; and its asymptotic power series expansion as $x \to \infty$ then given by the negative of the associated Taylor series to the left, $\sum_{m < m_0} \overset{\vee}{f}(m) x^m$.

But even in examples 1a and 1b, this formulation is too opportunistic to be correct. In example 1a we took $m_0 = 0$ since the Taylor series for f near $x = 0$ had a constant term and this worked; but in example 1b we needed to take $m_0 = 1$ and include the $m = 0$ term in the Taylor series to the left in order to capture the correct convergence of $\tan^{-1}(x)$ to $\frac{\pi}{2}$ as $x \to \infty$. Yet there is no apriori reason to justify this otherwise-arbitrary re-allocation of the $m = 0$ term. Indeed if we amended slightly to consider instead $f(x) := \tan^{-1}(x) - \frac{\pi}{4}$ then we would have to allocate part of the $m = 0$ coefficient to the Taylor series to the right and part of it to the Taylor series to the left.

In fact, as this last observation suggests, the problems with our naive conjecture as it stands go deeper than the question of where to put the bifurcation-point between what are terms to the "right" and what are terms to the "left". The next example illustrates this.

Example 1c [$f(x) = \frac{1+x^2}{1+x}$]: The function $f(x) = \frac{1+x^2}{1+x}$ is still smooth on $[0, \infty)$ and has $f(x) = \frac{1}{1+x} + x^2 \cdot \frac{1}{1+x}$ so that, from example 1a, we have $\overset{\vee}{f}(s) = \cos(\pi s) + \cos(\pi(s-2)) = 2 \cdot \cos(\pi s)$. Thus $\overset{\vee}{f}(m) = 2 \cdot (-1)^m$ for all $m \in \mathbb{Z}$.

But for x near 0, we have $f(x) = (1+x^2) \cdot \{1 - x + x^2 - x^3 + \ldots\} = 1 - x + 2x^2 - 2x^3 + 2x^4 - \ldots$; while for $x \to \infty$ we have $f(x) = \{\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} - \ldots\} + x^2 \cdot \{\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} - \ldots\} = x - 1 + 2\frac{1}{x} - 2\frac{1}{x^2} + 2\frac{1}{x^3} - \ldots$.

Thus there is no bifurcation point, m_0, under which $\sum_{m \geq m_0} \overset{\vee}{f}(m) x^m$ represents the power series expansion for $f(x)$ near 0; nor any such m_0 under which $\sum_{m < m_0} \overset{\vee}{f}(m) x^m$ represents the negative of the power series expansion for $f(x)$ as $x \to \infty$. Rather, we see that for degrees $m = 0$ and $m = 1$ the values of $\overset{\vee}{f}(0)$ and $\overset{\vee}{f}(1)$ *simultaneously* combine contributions from the power series expansions for $f(x)$ near 0 and near ∞.

By considering $f(x)$ of the form $\frac{p(x)}{1+x}$ for a more general polynomial we could in turn replicate this phenomenon at an arbitrarily large range of non-negative m-values. And likewise, by considering $f(x)$ of the form $\frac{p(\frac{1}{x})}{1+x}$ we could replicate it equally in a large range of negative m-values.

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[2] deduced as shown in [XI] either from $\overset{\vee}{f}(s) = -\mathcal{M}[f](-s) \cdot \frac{\sin(2\pi s)}{2\pi}$ or via the rules developed in that paper for building this correct canonical form from basic cases

What remains true in all such cases, however, is that for every $m \in \mathbb{Z}$ we have

$$\overset{\vee}{f}(m) = \left\{ \begin{array}{c} \text{coefficient of } x^m \text{ in the} \\ \\ \text{power series expansion} \\ \\ \text{for } f(x) \text{ near } 0 \end{array} \right\} - \left\{ \begin{array}{c} \text{coefficient of } x^m \text{ in the} \\ \\ \text{power series expansion} \\ \\ \text{for } f(x) \text{ as } x \to \infty \end{array} \right\} . \quad (3)$$

Let us state this as a precise conjecture.

Conjecture 1: *Suppose* f *is a smooth function on* $(0, \infty)$ *and suppose that for* x *near* 0, f *has the power series expansion* $\sum_{m=m_0}^{\infty} f_0(m)x^m$ *for some integer* m_0; *and that as* $x \to \infty$, f *has the power series expansion* $\sum_{m=-\infty}^{m_\infty} f_\infty(m)x^m$ *for some integer* m_∞. *Here, the power series of which* $\{f_0(m)\}_{m=m_0}^{\infty}$ and $\{f_\infty(m)\}_{m=-\infty}^{m_\infty}$ *are the coefficient sequences may be Taylor or Laurent series with some finite or infinite radius of convergence*, R; *or may merely be asymptotic expansions not convergent for any* x (so $R = 0$). *Then if* f *is integrable in a generalised Césaro sense on* $[0, \infty)$ *it has a canonical Taylor-coefficient function*, $\overset{\vee}{f}(s)$ $(s \in \mathbb{C})$, *and this satisfies that*

$$\overset{\vee}{f}(m) = f_0(m) - f_\infty(m) \quad \text{for all} \quad s = m \in \mathbb{Z}\,. \quad (4)$$

Comments: (a) [Poles and generalised Césaro interpretation]: In [XI] and so far here we have generally considered $f(x)$ smooth and classically integrable on $[0, \infty)$; so that f is non-singular near 0 and decays to zero sufficiently rapidly as $x \to \infty$.

As the examples of $f(x) = \frac{p(x)}{1+x}$ and $f(x) = \frac{p(\frac{1}{x})}{1+x}$ just considered suggest, however, there is no reason to avoid poles of finite order[3] at $x = 0$ and $x = \infty$. Such poles do not disturb the logic of the conjecture - and nor for that matter would any poles at finite points on $(0, \infty)$. All they do is necessitate that we expand our convergence framework to work in terms of generalised Césaro rather than purely classical convergence.

Then any pole at 0 (if $m_0 < 0$) or ∞ (if $m_\infty > 0$) or in-between leaves f Césaro-integrable on $[0, \infty)$. This allows calculation of $\mathcal{M}[f](-s)$ in generalised Césaro terms for generic $s \in \mathbb{C}$, and hence the canonical Taylor-coefficient function $\overset{\vee}{f}(s) := -\mathcal{M}[f](-s) \cdot \frac{\sin(2\pi s)}{2\pi}$ (from [XI]) exists, and the conjecture remains well-posed.

(b) [Nomenclature]: Since we are now allowing power series for $f(x)$ near 0 and ∞ which may be Taylor series or may be Laurent series; or which may only be asymptotic series with $R = 0$, we make one change in terminology.

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[3] i.e. m_0 and m_∞ finite

From now on we will refer to $\overset{\vee}{f}(s)$ as the TLA-coefficient function of f, rather than as its Taylor-coefficient function. Obviously, TLA here stands for Taylor-Laurent-asymptotic, so this new nomenclature serves as a reminder of the more general domain of applicability which now pertains for conjecture 1.

(c) [$\overset{\vee}{f}$ as global quantity]: The balance in the roles of 0 and ∞ in conjecture 1 reflects the fact that the TLA-coefficient function, $\overset{\vee}{f}(s)$, is a *global* quantity, even where f is as smooth, non-singular and rapidly-decaying as could be wished for.

This may seem surprising given that, in [XI], we started examining it based just on traditional Taylor-coefficients $\overset{\vee}{f}(m) = \frac{f^{(m)}(0)}{m!}$ ($m \in \mathbb{Z}_{\geq 0}$), which obviously involve only the value of f and its derivatives at 0 and which are therefore quintessentially local, depending only on f in a neighbourhood of 0.

However, as soon as we consider $s \notin \mathbb{Z}$, this global character becomes clear. For such general s we have $\overset{\vee}{f}(s) = -\mathcal{M}[f](-s) \cdot \frac{\sin(2\pi s)}{2\pi}$ and since $\mathcal{M}[f](-s) = \int_0^\infty x^{-s-1} f(x)\,\mathrm{d}x$ is an integral over all of $[0, \infty)$, it follows that $\overset{\vee}{f}(s)$ in general depends on f on the whole of this domain and, in particular, is as much dependent on its behaviour near $x = \infty$ as it is on its behaviour near $x = 0$.

Another way of viewing this, at least in the smooth and rapidly-decaying case, is that $\overset{\vee}{f}(s)$ in some sense represents $\frac{1}{s!} \cdot \left(\frac{d}{dx}\right)^s [f](0)$. Whether the fractional derivative $\left(\frac{d}{dx}\right)^s [f](0)$ is understood using the definitions of fractional calculus (see e.g. [IX]) or via Fourier theory (see e.g. [VII]), it is certainly non-local whenever $s \notin \mathbb{Z}$. It only becomes local at $s = m \in \mathbb{Z}_{\geq 0}$ because the associated integrals in these definitions then become distributional versions of the delta-function and its derivatives.

(d)[Care with local-to-global inference]: All of this, however, raises a question. In [XI], and even in the examples so far in this section, we have generally built $\overset{\vee}{f}(s)$ just out of a general formula for $\overset{\vee}{f}(m)$ as $\frac{f^{(m)}(0)}{m!}$, focusing exclusively on f near 0 (albeit with the assistance of some additional tools like lemmas 1a-1c in [XI]).

For the most part we never considered f near ∞ in constructing $\overset{\vee}{f}(s)$, nor worked via its general Mellin-transform definition (indeed in section 4 of[XI] we devoted some time to showing how to work in the reverse direction and calculate Mellin transforms using TLA-coefficient functions and Taylor-series-to-the-left methods). Why then did we not encounter problems?

The answer is that in most instances, the examples we considered were smooth and rapidly-decaying as $x \to \infty$. In such circumstances there is a clean bifurcation, with only non-negative powers of x ($m \geq 0$) in the power series for f near 0 and only negative powers ($m < 0$) in the power series for f as $x \to \infty$.

In [XI] this was done (e.g. in examples such as $f(x) = \mathrm{e}^{-x^n}$ or $f(x) = \frac{1}{1+x^n}$)

in order to ensure classical integrability in the initial testing of conjectures 1-3 (regarding integrals on $[0,\infty)$), but it serendipitously ensured that there was no contribution from $f_\infty(m)$ to $\overset{\vee}{f}(m)$ for any $m \geq 0$ and therefore no error in our deduction of the functional form of $\overset{\vee}{f}(s)$ just from $f_0(m)$.

In example 1a here (with $f(x) = \frac{1}{1+x^2}$) it likewise guaranteed that there was a clean bifurcation, with no simultaneous intermingling of contributions to any $\overset{\vee}{f}(m)$ from both $f_0(m)$ and $f_\infty(m)$. We could thus focus exclusively on $x = 0$ to intuitively derive the form of $\overset{\vee}{f}(m)$ from $f_0(m)$ for $m \geq 0$; and thence deduce $f_\infty(m)$ from $\overset{\vee}{f}(m)$ using this formula, thereby giving us the power series for $f(x)$ as $x \to \infty$.

This sort of intuition and deduction is the essence of what, in this set of papers, we have called local-to-global inference.

In examples 1a-1c here, we could use algebraic manipulations to verify that the resulting power series for $f(x)$ as $x \to \infty$ were correct. This sort of reasoning should, however, remain feasible - allowing us to deduce the power series for $f(x)$ as $x \to \infty$ - irrespective of whether there is an algebraic method for verifying $\{f_\infty(m)\}$ directly. Indeed conjecture 1 precisely captures the correct general relationship between $\overset{\vee}{f}(m)$, $f_0(m)$ and $f_\infty(m)$ required to facilitate such inference.

While conjecture 1 thus facilitates rather than undermines the feasibility of such local-to-global inference, the intermingled relationship of equation 4 does, however, indicate that we need to be more careful than we have been in most examples so far, when undertaking such inference in general. And the global character of $\overset{\vee}{f}(s)$ also means that we need to take account of this intermingling of behaviour near both 0 and ∞ when intuiting the right canonical form for $\overset{\vee}{f}(s)$ in the first place.

We have seen this in examples like $f(x) = \mathrm{e}^{-\frac{1}{1+x^2}}$ and $f(x) = \ln(\Gamma(1+x))$ in [XI]; and like $f(x) = \tan^{-1} x - \frac{\pi}{4}$ or $f(x) = \frac{p(x)+p(\frac{1}{x})}{1+x^2}$ here. As soon as we get away from smooth and rapidly-decaying functions with a clear bifurcation point in m, we get intermingling of contributions from $x = 0$ and $x = \infty$ in $\overset{\vee}{f}(m)$ and care needs to be taken in prising apart and attributing components of $\overset{\vee}{f}(m)$ to $f_0(m)$ and $f_\infty(m)$ separately.

At the same time, since this relates to the appearance of power-divergences in $f(x)$ at 0 and ∞ and since, as noted in comment (a), these are easily-handled within the generalised Césaro convergence framework, this framework should now become the natural setting of both conjecture 1 here and all the preceding work in [XI] (including the definition and calculation of Mellin transforms).

In the fourth paper in this set we give an elegant demonstration of the capacity for local-to-global inference using conjecture 1, even where all these issues - of power-divergences, intermingling and the need for generalised Césaro interpretation - are present. For now, however, we turn to demonstrating merely that it can be used for more than the trivial examples considered so far in this section.

3 A non-trivial example

In [IV] we considered a problem inspired by a calculation in [1, section 10] - can we find an explicit, simply-defined function, $f(x)$, with the prescribed (nowhere-convergent) asymptotic series $\sum_{j=0}^{\infty} B_{j+1}x^j = \sum_{j=0}^{\infty}(-1)^j \cdot (j+1) \cdot \zeta(-j)x^j$ as $x \to 0$?

In [IV] our focus was on Césaro arrays, so we simply took a proposed such function, $f(x) = -\frac{1}{x} + \sum_{j=1}^{\infty} \frac{1}{(jx+1)^2}$, and verified after-the-fact using Césaro arrays that it does indeed have the stipulated asymptotic power series for x near 0.

We did not explain, however, how we came up with this proposal for $f(x)$ in the first place. The answer is that it comes from Taylor-series-to-the-left analysis using conjecture 1 as follows.

It is prescribed that $f_0(m) = (-1)^m \cdot (m+1) \cdot \zeta(-m) = \cos(\pi m) \cdot (m+1) \cdot \zeta(-m)$ for $m \in \mathbb{Z}_{\geq 0}$, and if we look for a candidate, $f(x)$, which decays to zero as $x \to \infty$, then this becomes the formula for $\overset{\vee}{f}(m)$ also. We thus try taking

$$\overset{\vee}{f}(s) = \cos(\pi s) \cdot (s+1) \cdot \zeta(-s)$$

in general.

Since, by prescription, $f_0(m) = 0$ for all $m \in \mathbb{Z}_{<0}$, it follows by conjecture 1 that the Taylor series to the left for f, given by $\sum_{m=-\infty}^{-1} \overset{\vee}{f}(m)x^m$, must then represent the negative of the power series for $f(x)$ as $x \to \infty$.

Since $\overset{\vee}{f}(-1) = \lim_{\epsilon \to 0} \overset{\vee}{f}(-1+\epsilon) = \lim_{\epsilon \to 0} \cos(\pi(-1+\epsilon)) \cdot \epsilon \cdot \zeta(1-\epsilon) = 1$, we thus have that, for x large,

$$\begin{aligned}
f(x) &= -\frac{1}{x} + \sum_{n=2}^{\infty}(-1)^n \cdot (n-1) \cdot \zeta(n) \cdot \frac{1}{x^n} \\
&= -\frac{1}{x} + \zeta(2) \cdot \frac{1}{x^2} - 2\zeta(3) \cdot \frac{1}{x^3} + 3\zeta(4) \cdot \frac{1}{x^4} - \ldots \\
&= \left\{ \begin{array}{c} -\left\{\frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} - \frac{3}{x^4} + \ldots\right\} \\ \\ +\left\{\frac{(\zeta(2)-1)}{x^2} - 2 \cdot \frac{(\zeta(3)-1)}{x^3} + 3 \cdot \frac{(\zeta(4)-1)}{x^4} - \ldots\right\} \end{array} \right\} \\
&= \left\{ \begin{array}{c} -\frac{d}{dx}\left\{\ln x + \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} - \frac{1}{x^4} + \ldots\right\} \\ \\ -\frac{d}{dx}\left\{\frac{(\zeta(2)-1)}{x} - \frac{(\zeta(3)-1)}{x^2} + \frac{(\zeta(4)-1)}{x^3} - \ldots\right\} \end{array} \right\} .
\end{aligned}$$

Now we have that $\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x^3}-\frac{1}{x^4}+\ldots=\frac{1}{x+1}$; and $\zeta(n)-1=\sum_{j=2}^{\infty}\frac{1}{j^n}$ so that

$$\begin{aligned}
\sum_{n=2}^{\infty}(-1)^n\frac{(\zeta(n)-1)}{x^{n-1}} &= x\cdot\sum_{n=2}^{\infty}(-1)^n\sum_{j=2}^{\infty}\frac{1}{j^n x^n}\\
&= x\cdot\sum_{j=2}^{\infty}\sum_{n=2}^{\infty}\frac{(-1)^n}{j^n x^n}\\
&= x\cdot\sum_{j=2}^{\infty}\frac{1}{j^2x^2}\cdot\left\{1-\frac{1}{jx}+\frac{1}{j^2x^2}-\ldots\right\}\\
&= x\cdot\sum_{j=2}^{\infty}\frac{1}{j^2x^2}\cdot\frac{jx}{jx+1}=\sum_{j=2}^{\infty}\frac{1}{j(jx+1)} \quad .
\end{aligned}$$

Thus we have that

$$f(x)=-\frac{d}{dx}\left\{\ln x+\sum_{j=1}^{\infty}\frac{1}{j(jx+1)}\right\}=-\frac{1}{x}+\sum_{j=1}^{\infty}\frac{1}{(jx+1)^2}$$

and since $\sum_{j=1}^{\infty}\frac{1}{(jx+1)^2}$ is in fact classically convergent for arbitrary $x>0$, so this choice of $f(x)$ is well-defined not just for large x but on all of $(0,\infty)$.

We thus see that conjecture 1 and Taylor-series-to-the-left methodology is what lay behind the derivation of the solution function with the prescribed asymptotic series we analysed in [IV].

Comment: This solution function satisfies that $\lim_{x\to 0}f(x)=-\frac{1}{2}=B_1$, as it should. This is the case because the $-\frac{1}{x}$ term in $f(x)$ cancels a $\frac{1}{x}$-divergence as $x\to 0$ which, as we showed in [IV], arises in $\sum_{j=1}^{\infty}\frac{1}{(jx+1)^2}$.

However, as $x\to\infty$ it is easy to see that $\sum_{j=1}^{\infty}\frac{1}{(jx+1)^2}\sim\zeta(2)\cdot\frac{1}{x^2}$, so the $-\frac{1}{x}$ term remains uncancelled as $x\to\infty$ and it is this which leads to the fact that $\overset{\vee}{f}(-1)=1$ (we have $f_0(-1)=0$ and $f_\infty(-1)=-1$ so that, by conjecture 1, $\overset{\vee}{f}(-1)=f_0(-1)-f_\infty(-1)=1$).

It is nevertheless worth noting that $\overset{\vee}{f}(s)$ is in fact unaffected by the inclusion or omission of this stand-alone $-\frac{1}{x}$ term. If we were to omit it and consider simply $g(x):=\sum_{j=1}^{\infty}\frac{1}{(jx+1)^2}$, then we would have $g_0(-1)=1$ and $g_\infty(-1)=0$ and would therefore still have $\overset{\vee}{g}(-1)=1$; and since $s=-1$ is the only point at which $\overset{\vee}{f}(s)$ and $\overset{\vee}{g}(s)$ could conceivably differ, we thus have in general that $\overset{\vee}{g}(s)=\overset{\vee}{f}(s)$ as functions on $\mathbb{C}$.

This is in fact an example of a general phenomenon (which we discussed in the case of $f(x)=\mathrm{e}^{-\frac{1}{1+x^2}}-1$ and elsewhere in section 3 in [XI] and also saw when we considered the variants $f(x) = \tan^{-1} x$ and $g(x) = \tan^{-1} x - \frac{\pi}{4}$ in example 1b), namely that:

Result 1: *If* $g(x) = f(x) + a_m x^m$, $m \in \mathbb{Z}$, *then* $\overset{\vee}{g}(s) = \overset{\vee}{f}(s)$ *as functions on* $\mathbb{C}$.

This follows since the addition of the term $a_m x^m$ just increases $g_0(m)$ and $g_\infty(m)$ by a_m vis-a-vis the corresponding quantities for f and thus, by conjecture 1, leaves $\overset{\vee}{g}(m) = \overset{\vee}{f}(m)$; while there is obviously no change in moving from $\overset{\vee}{f}(s)$ to $\overset{\vee}{g}(s)$ for any $s \neq m$. In fact, Result 1 holds not just for the addition of a stand-alone term of the form $a_m x^m$, $m \in \mathbb{Z}$, but for addition of a general stand-alone term (or finite collection of such terms) of the form $a_\rho x^\rho$, $\rho \in \mathbb{C}$. The proof traces similar lines to those followed in the derivations in section 4 in [XI] - by considering first $\rho \in \mathbb{Q}$, thence $\rho \in \mathbb{R}$ by passing to limits, and finally arbitrary $\rho \in \mathbb{C}$ by analytic continuation - but we omit details here.

4 Two final examples regarding conjecture 1

We conclude this paper with two final examples. The first extends the interpretation of conjecture 1 beyond the boundaries encountered in examples thus far. The second points to the need to be careful in splitting values of $\overset{\vee}{f}(m)$, $m \in \mathbb{Z}$ into components $f_0(m)$ and $f_\infty(m)$ - and discusses why this is relevant for any efforts towards a calculus of TLA-coefficient functions describing how they behave under combination by product, quotient, composition etc.

Example 2a [$f(x) = \ln(1+x)$]: For small x we have $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$, so $f_0(m) = \frac{(-1)^{m+1}}{m}$ for $m \in \mathbb{Z}_{>0}$ and $f_0(m) = 0$ for all $m \in \mathbb{Z}_{\leq 0}$.

And for large x we have $\ln(1+x) = \ln x + \ln(1+\frac{1}{x}) = \ln x + \frac{1}{x} - \frac{1}{2}\frac{1}{x^2} + \frac{1}{3}\frac{1}{x^3} - \ldots$. Thus $f_\infty(m) = \frac{(-1)^m}{m}$ for $m \in \mathbb{Z}_{\leq -1}$ and $f_\infty(m) = 0$ for all $m \in \mathbb{Z}_{>0}$, while $f(x)$ also exhibits a logarithmic divergence as $x \to \infty$ which does not immediately fit into the framework regarding $f_\infty(m)$ which we have developed so far.

By conjecture 1 we see that we have

$$\overset{\vee}{f}(m) = \frac{(-1)^{m+1}}{m} \quad \text{for all} \quad m \in \mathbb{Z} \setminus \{0\}$$

suggesting that we should have

$$\overset{\vee}{f}(s) = \frac{\cos(\pi s)}{s}$$

in general.

We see at once that this has a singularity at $s = 0$, namely a simple pole with residue -1, meaning that $\overset{\vee}{f}(s) \approx -\frac{1}{s} + \mathit{convgt}$ for s near 0. But this is not a problem! in fact it represents $\overset{\vee}{f}(s)$ correctly capturing the log-divergence which exists in $f(x)$ as $x \to \infty$.

We shall explain this rigorously in the next paper, based on our canonical form for $\overset{\vee}{f}(s)$ and the generalised Césaro behaviour of Mellin transforms. But for now it is easy to see heuristically how this interpretation augments the existing framework.

After all, we know that $\ln x$ is the anti-derivative of $\frac{1}{x}$, so it is appropriate that its presence should turn up on the s-side in the behaviour of $\overset{\vee}{f}(s)$ at $s = 0$ (in the same way that the anti-derivative of x^ρ will turn up in the value of $\overset{\vee}{f}(s)$ at $s = \rho + 1$). And the fact that its residue at $s = 0$ is -1 reflects that it corresponds to an $\ln x$-divergence with coefficient 1, but one which occurs as $x \to \infty$, so that it appears with a negative sign under conjecture 1.

We thus see that conjecture 1, suitably interpreted, continues to hold in the presence of logarithmic divergences. A function having a divergence of the form $a \ln x$ as $x \to 0$ and $b \ln x$ as $x \to \infty$ has a simple pole in $\overset{\vee}{f}(s)$ with residue $(a - b)$ at $s = 0$.

And this in turn extends, first, to higher powers - so that if $f(x)$ has a divergence of the form $a(\ln x)^m$ as $x \to 0$ and has a divergence of the form $b(\ln x)^m$ as $x \to \infty$, then $\overset{\vee}{f}(s)$ has a pole of order m with residue $(a - b)$ at $s = 0$; and then also to power-log divergences - so that if $f(x)$ has a divergence of the form $ax^\rho(\ln x)^m$ as $x \to 0$ and has a divergence of the form $bx^\rho(\ln x)^m$ as $x \to \infty$, then $\overset{\vee}{f}(s)$ has a pole of order m with residue $(a - b)$ at $s = \rho$ (and thus $\overset{\vee}{f}(s) \approx \frac{(a-b)}{(s-\rho)^m}$ for s near ρ).

Example 2b [$f(x) = \mathbf{e}^{-x}$]: For this example we return to the simplest of cases, the very first one we considered in [XI], namely $f(x) = \mathrm{e}^{-x}$.

As discussed in [XI], this has $\overset{\vee}{f}(s) = \frac{\cos(\pi s)}{s!}$ and so $\overset{\vee}{f}(m) = \frac{\cos(\pi m)}{m!}$ for all $m \in \mathbb{Z}_{<0}$. Since $f_0(m) = 0$ for all $m \in \mathbb{Z}_{<0}$ it is tempting to assume that the coefficients $\overset{\vee}{f}(m)$ for $m \in \mathbb{Z}_{<0}$ should all be attributed to $f_\infty(m)$; or alternatively, since $\frac{\cos(\pi m)}{m!}$ is always zero when $m \in \mathbb{Z}_{<0}$, to postulate that it doesn't matter whether these coefficients are attached, in whole or in part, to $f_0(m)$ or to $f_\infty(m)$. After all, why should it matter how we divide up zero between two pools - one giving the power series for $f(x)$ near 0, and the other the power series for $f(x)$ near ∞?

The answer is that it does matter. First, we have seen in conjecture 1 from [XI] that we cannot ignore zeros of $\overset{\vee}{f}(s)$. Césaro integrability of f on $[0, \infty)$ is bound up in whether $\overset{\vee}{f}(-1) = 0$; and since then $\int_0^\infty f(x)\,\mathrm{d}x = \overset{\vee}{f}'(-1)$, it matters whether such zeros are of order 1 or more.

Secondly, it does matter how we allocate such zeros. One reason is the following lemma:

Lemma 1a: *Suppose we have two functions, f and g, with TLA-coefficient functions, $\overset{\vee}{f}$ and $\overset{\vee}{g}$, each with its component pieces - $\{f_0(m)\}_{m=m_0}^{\infty}$ and $\{f_\infty(m)\}_{m=-\infty}^{m_\infty}$ on the one hand; and $\{g_0(m)\}_{m=m_0'}^{\infty}$ and $\{g_\infty(m)\}_{m=-\infty}^{m_\infty'}$ on the other. Then $h(x) := f(x) \cdot g(x)$ has TLA-coefficient function satisfying*

$$\overset{\vee}{h}(m) = h_0(m) - h_\infty(m) \tag{5}$$

where

$$h_0(m) = \sum_{j=-\infty}^{\infty} f_0(j) \cdot g_0(m-j) \quad \text{and} \quad h_\infty(m) = \sum_{j=-\infty}^{\infty} f_\infty(j) \cdot g_\infty(m-j)\ . \tag{6}$$

For any given $m \in \mathbb{Z}$, note that the sums giving $h_0(m)$ and $h_\infty(m)$ will in fact be finite sums.

This follows immediately from the meaning of the components h_0 and h_∞ in terms of the power series for $h(x)$ separately near 0 and near ∞; and the corresponding interpretations for the constituent functions f and g.

The key point is that calculating the pieces, h_0 and h_∞, for the product (and hence $\overset{\vee}{h}(m)$, at least for $m \in \mathbb{Z}$) depends critically on how we split $\overset{\vee}{f}(m)$ and $\overset{\vee}{g}(m)$ into their respective pieces, including attribution of pieces which are themselves zero!

As we develop further the "calculus" for behaviour of TLA-coefficient functions over the remainder of this set of papers, we will see a number of examples where retention of zeros and careful attribution of them will be crucial in facilitating some otherwise-difficult computations. This is especially so in instances where we are able to reduce apparently continuous calculations, such as for a derivative like $\overset{\vee}{h}'(-1)$, to essentially discrete, combinatorial calculations. We will conclude this paper with one such instance after quickly completing our discussion of example 2b.

Returning to the question we started with for this example - for $f(x) = \mathrm{e}^{-x}$ the correct answer is that the negative TLA-coefficients, $\frac{\cos(\pi m)}{m!}$, $m \in \mathbb{Z}_{<0}$, should be attributed *entirely* to $f_0(m)$, not $f_\infty(m)$. Let us explain why heuristically.

The reason is that e^{-x} is *Schwartzian* as $x \to \infty$. Now suppose, say, the $m = -1$ TLA-coefficient of f were attributed to f_∞. Since $\frac{1}{(-1+\epsilon)!} \approx \epsilon$, the zero at $f_\infty(-1)$ would be of order 1. Now suppose we were to multiply $f(x)$ by $g(x) = \ln(1+x)$, which we saw in example 2a has a pole of order 1 attributable entirely to g_∞ at $m = 0$, with residue 1. Then lemma 1a would imply that $h(x) := f(x) \cdot g(x)$ would have $h_\infty(-1) = 1$. But this would imply that $h(x)$ has a power series approximation as $x \to \infty$ containing a term of the form $\frac{1}{x}$, when in fact $\mathrm{e}^{-x} \cdot \ln(1+x)$ remains Schwartzian as $x \to \infty$, a contradiction. It follows that $\overset{\vee}{f}(-1)$ must actually be attributable entirely to f_0 rather than f_∞, and a similar argument shows the same also for $\overset{\vee}{f}(m)$, $m \in \mathbb{Z}_{\leq -2}$.

Comment: In the argument just given we seem to be getting onto somewhat delicate ground.

This is because, while $\overset{\vee}{f}(s)$ is defined for all continuous $s \in \mathbb{C}$, the quantities f_0 and f_∞ are really only defined for discrete values $s = m \in \mathbb{Z}$. Yet we have argued by considering $\overset{\vee}{f}$ and $\overset{\vee}{g}$ at $s = -1+\epsilon$ while still continuing the attribution of these values to, in this case, f_∞ and g_∞ in order to argue that the combination of the order 1 zero (in $\overset{\vee}{f}$ at $s = -1$) and pole (in $\overset{\vee}{g}$ at $s = 0$) must lead to a $\frac{\pm 1}{x}$ term we can still attribute to the power series for $h(x)$ as $x \to \infty$.

The precise justification for this heuristic reasoning still requires some clarification - we will provide this later in this section. But in any case, for now we apply the same sort of argument in our promised final example - which also serves to confirm the attribution of all of the terms $\overset{\vee}{f}(m)$, $m \in \mathbb{Z}$, for $f(x) = \mathrm{e}^{-x}$ to f_0 rather than f_∞.

Final sample calculation - evaluation of $\int_0^\infty \mathbf{e}^{-nx}\,\mathbf{d}x$: For $h(x) = \mathrm{e}^{-nx}$ we have that $\int_0^\infty h(x)\,\mathrm{d}x = \frac{1}{n}$, since $\overset{\vee}{h}(s) = \cos(\pi s)\frac{n^s}{s!}$ and this has $\overset{\vee}{h}(-1) = 0$ and $\overset{\vee}{h}'(-1) = -\frac{1}{n}$. Let us test this (admittedly trivial) calculation using lemma 1a with $f(x) = \mathrm{e}^{-kx}$ and $g(x) = \mathrm{e}^{-(n-k)x}$.

Then

$$\overset{\vee}{f}(s) = \cos(\pi s)\frac{k^s}{s!} \quad \text{and} \quad \overset{\vee}{g}(s) = \cos(\pi s)\frac{(n-k)^s}{s!}$$

and if the coefficients for $s = m \in \mathbb{Z}_{<0}$ were attributable to f_∞ and g_∞, the calculation of $\overset{\vee}{h}'(-1)$ and hence $\int_0^\infty h(x)\,\mathrm{d}x$ using lemma 1a along the lines applied above would fail.

By contrast, if all of the $\overset{\vee}{f}(m)$ are attributable to f_0 and likewise for g, it follows in lemma 1a that in fact we have $h_\infty(m) = 0$ for all m, while

$$\begin{aligned} \overset{\vee}{h}(m) &= h_0(m) = \sum_{j=-\infty}^{\infty} f_0(j) g_0(m-j) \\ &= \sum_{j=-\infty}^{\infty} \cos(\pi j)\frac{k^j}{j!} \cdot \cos(\pi(m-j))\frac{(n-k)^{m-j}}{(m-j)!} \quad . \end{aligned}$$

It follows at once that $\overset{\vee}{h}(-1) = 0$ and, taking $s = -1 + \epsilon$ and using the same heuristic reasoning as just invoked, we have that

$$\begin{aligned}
\overset{\vee}{h}(-1+\epsilon) &= \sum_{j=-\infty}^{\infty} \cos(\pi j)\frac{k^j}{j!} \cdot \cos(\pi(-j-1+\epsilon))\frac{(n-k)^{-j-1+\epsilon}}{(-j-1+\epsilon)!} \\
&= \sum_{j=0}^{\infty} \cos(\pi j)\frac{k^j}{j!} \cdot (-1)^{-j-1}(-1)^j (n-k)^{-j-1} j!\,\epsilon + O(\epsilon^2) \\
&= -\frac{1}{(n-k)} \cdot \left\{ 1 - \frac{k}{(n-k)} + \frac{k^2}{(n-k)^2} - \dots \right\} \epsilon + O(\epsilon^2) \\
&= -\frac{1}{n}\,\epsilon + O(\epsilon^2)
\end{aligned}$$

so that $\overset{\vee}{h}'(-1) = -\frac{1}{n}$ and we obtain the correct value for $\overset{\vee}{h}'(-1)$ and hence for $\int_0^\infty \mathrm{e}^{-nx}\,\mathrm{d}x$.

This gives further confirmation of the correctness of our attribution of all the $\overset{\vee}{f}(m)$ entirely to $f_0(m)$ (and likewise for g and h), and also provides a nice initial demonstration of how retaining these zeros in this attribution is crucial; and is in turn critical in working via $s = -1+\epsilon$ and lemma 1a to turn the resulting calculation of $\overset{\vee}{h}'(-1)$ into an essentially discrete, combinatorial calculation in a way that will be very useful in more complex calculations in future papers.

Final Comments: (a) We remarked in our last comment that the precise justification for our heuristic reasoning in moving off discrete m to continuous s in the manner we have done is still unclear, but this is not quite fair. The logic is actually quite straightforward. Bringing it together it runs as follows.

From the correct attributions in $\overset{\vee}{f}$ and $\overset{\vee}{g}$ at discrete $m \in \mathbb{Z}$, lemma 1a gives us the correct formula for $\overset{\vee}{h}$ at all such m. Then moving from discrete m to continuous s in $\overset{\vee}{h}$ (rather than in h_0 or h_∞ per se) is no more than we have been investigating at length in [XI], and requires only that this be done using the techniques developed there to come up with the right gauge-choice for $\overset{\vee}{h}(s)$ which matches its canonical form as $-\mathcal{M}[h](-s) \cdot \frac{\sin(2\pi s)}{2\pi}$.

Since lemma 1a tells us that

$$\overset{\vee}{h}(m) = \sum_{j=-\infty}^{\infty} f_0(j) \cdot g_0(m-j) - \sum_{j=-\infty}^{\infty} f_\infty(j) \cdot g_\infty(m-j)$$

it follows immediately that this move from m to s should give

$$\overset{\vee}{h}(s) = \sum_{j=-\infty}^{\infty} f_0(j) \cdot g_0(s-j) - \sum_{j=-\infty}^{\infty} f_\infty(j) \cdot g_\infty(s-j) \tag{7}$$

and this is in fact all that we have done in the example calculations using exponentials just undertaken (where life was further simplified by having f_∞ and

g_∞ both identically zero).

(b) Nonetheless, while this justifies the reasoning we have invoked, it leads to the observation that equation 7 does not treat f and g equally.

One function (namely f) remains evaluated only at discrete j, while the other (namely g) is now evaluated at $s - j \notin \mathbb{Z}$. Since the roles of f and g in a product $h(x) = f(x) \cdot g(x)$ are interchangeable, it follows that we must be able to interchange the roles of f and g in equation 7, but it is not self-evident why this should give the same result for $\overset{\vee}{h}(s)$, especially in more complicated examples where f and g are completely different types of functions (such as $f(x) = \mathrm{e}^{-x}$, $g(x) = \ln(1 + x)$).

In fact, the capacity to make the *choice* of which function to take as f and which as g in such a product, can be viewed as a bonus. It may allow us a choice which, for example, harnesses the zeros of $\overset{\vee}{f}$ or $\overset{\vee}{g}$ at discrete m in a way which facilitates computation of $\overset{\vee}{h}(s)$, where the opposite choice may perhaps lead to computational difficulties.

Indeed, one common issue we will encounter in such choices is whether the resulting calculation for $\overset{\vee}{h}$ is convergent or divergent, and this is the case even for the calculation with $f(x) = \mathrm{e}^{-kx}$ and $g(x) = \mathrm{e}^{-(n-k)x}$ we just performed.

In that case, the geometric series which resulted, namely $1 - \frac{k}{(n-k)} + \frac{k^2}{(n-k)^2} - \ldots$, is convergent provided $k \leq n - k$ (if $k = n - k$ it is still convergent to the correct value, but only under evaluation as a generalised Césaro sum). If $k > n - k$ the series becomes divergent, and in a way which is not Césaro-amenable[4]. But in this case we could reduce to the previous tractable case simply by reversing the roles of f and g in equation 7.

We will end up using lemma 1a and these sorts of calculations using equation 7 extensively in later papers in this set. They give us a calculus for TLA-coefficient functions under taking products as a first step in a general calculus for such functions, and they allow us to reduce many integrals to discrete combinatorial sums. In applying them, however, we will often need to make astute choices of f and g in order to enable computation.

5 Acknowledgements

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References

[I] R. Stone, *Introduction to generalised Césaro convergence I*, 2026

[II] R. Stone, *Introduction to generalised Césaro convergence II*, 2026

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[4] It is a geometric, not a Dirichlet, series and so would require a Borel convergence scheme

[III] R. Stone, *Introduction to generalised Césaro convergence III*, 2026

[IV] R. Stone, *Césaro Arrays I*, 2026

[V] R. Stone, *Césaro Arrays II*, 2026

[VI] R. Stone, *Césaro Arrays III*, 2026

[VII] R. Stone, *Root Identities I*, 2026

[VIII] R. Stone, *Root Identities II: Root identities for* ζ *- Part A*, 2026

[IX] R. Stone, *Root Identities II: Root identities for* ζ *- Part B*, 2026

[X] R. Stone, *Root Identities II: Root identities for* ζ *- Part C*, 2026

[XI] R. Stone, *Taylor series to the left I - Integration*, 2026

[1] H.M. Edwards, *Riemann's Zeta Function*, Academic Press, 1974