Taylor Series to the Left III - Taylor and Mellin transforms in a generalised Césaro framework

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February 5, 2026

Abstract

$$Oh\ it's\ Tommy\ this,\ an'\ Taylor\ that;\ \text{Césaro}\ do\ your\ part!$$
$$But\ it's\ \text{"}Thank\ you\ Mr\ Mellin\text{"}\ when\ the\ transforms\ need\ to\ start,$$
$$Yes\ at\ root\ it's\ Mr\ Taylor,\ with\ \text{Césaro}\ in\ his\ heart,$$
$$And\ it's\ \text{"}Thank\ you\ Mr\ Mellin\text{"}\ when\ the\ transforms\ need\ to\ start!$$

We consider the generalised Césaro framework and demonstrate that it contains a natural inner-product. This gives rise to a "Taylor transform" and we show that this is formally equivalent to the well-known Mellin transform. This establishes that the generalised Césaro convergence framework is in fact the natural setting for the Mellin transform. We demonstrate this in multiple ways - by showing how such a generalised Césaro perspective radically simplifies the expression of many results in the theory of Mellin transforms; and by using it to easily connect to, and then prove, the results relating to Taylor-series-to-the-left methods and TLA-coefficient functions developed in the first two papers ([XI] and [XII]) in this series.

1 Introduction

This is the third in a set of papers on the notion of TLA-coefficient functions and Taylor-series-to-the-left methods. In the first ([XI]) we introduced the motivation for these concepts, and showed how they relate to integration in general and, in particular, to the Mellin transform of a function $f(x)$.

In the second ([XII]) we in turn related the TLA-coefficient function, $\overset{\vee}{f}(s)$, to the behaviour of $f(x)$ not just for x near 0, but also simultaneously as $x \to \infty$.

In both papers, we considered many illustrative examples. And among these, we included a number showing that these Taylor-series-to-the-left methods - and the associated precise conjectures which we developed in [XI] and [XII] regarding the concepts of TLA-coefficient functions and their application to integrals, Mellin transforms and the asymptotic behaviour of f - all seem to hold not just in the domain of classical convergence, but equally well within the broader generalised Césaro convergence framework.

In this paper we tie all these ideas together and establish them on a rigorous footing. But to do so we begin, in section 2, by returning to an abstract consideration of the generalised Césaro framework itself, albeit motivated by a review of the core features we have delineated in the examples analysed in [XI] and [XII].

This leads us in section 2.1 to consider a natural inner-product, which only makes sense under this broader Césaro convergence perspective. We specify this inner-product and demonstrate its key properties, and then in turn we define the "Taylor transform" to which it naturally gives rise and show that it is formally equivalent to the well-known Mellin transform. In so doing we establish that the generalised Césaro convergence framework is in fact the appropriate setting for understanding the Mellin transform.

We demonstrate this in section 2.2 by showing how the adoption of this generalised Césaro perspective directly connects $\mathcal{M}[f](-s)$ to the form of the power series expansions for $f(x)$ near 0 and near ∞, and more generally to the presence of powers, x^{s_0}, in $f(x)$.

In particular, in section 2.3 we then show how this integrated generalised Césaro perspective allows us to prove the key conjecture we derived in [XII] regarding the TLA-coefficient function, $\overset{\vee}{f}(s)$ - namely that for $m \in \mathbb{Z}$, $\overset{\vee}{f}(m)$ is given by the value of the coefficient of x^m in the power series for $f(x)$ near 0, minus the coefficient of x^m in the power series for $f(x)$ near ∞.

In the same way, in section 2.3 we also convert all the other conjectures and observations of [XI] and [XII] into well-established results within the generalised Césaro framework; as we do with the associated processes of integration using Taylor-series-to-the left methods developed in [XI], and the methods of local-to-global inference developed in [XII].

In section 2.4 we then further show how the adoption of this integrated perspective removes or simplifies many of the technical constraints which beset the theory of the Mellin transform, $\mathcal{M}$, in its traditional treatment within the confines of classical convergence.

We summarise this consolidated set of results in section 3, where we also illustrate this now-integrated framework of Mellin transforms, generalised Césaro convergence and Taylor-series-to-the-left methods with a number of additional examples, and with a verification that the well-known core properties of $\mathcal{M}$ continue to hold within the generalised Césaro domain.

As part of this we explore how the connection to TLA-coefficient functions now gives a new approach for attempting the calculation of complicated Mellin transforms (e.g. of complicated products of functions) by working instead via these TLA-coefficient functions and using the algebraic and combinatorial methods we have developed for them.

We conclude in section 3.1 with one final big-picture observation regarding the scope for mimicking our approach here - for Mellin transforms and generalised Césaro convergence - and applying it to the case of other transforms and the generalised convergence schemes naturally associated to them.

1.1 Final notes

We thank Prof Tommy Atkins of Redcoat University for many helpful comments and suggestions (pers. comm.), and we dedicate this paper to him and to his compatriots, with thanks for their enormous contributions over a long period of time.

2 Generalised Césaro convergence, inner products, and Taylor and Mellin transforms

In Fourier transform theory, the central constituents are oscillatory exponential functions of the form $\mathrm{e}^{i\xi x}$, $\xi \in \mathbb{R}$. These are eigenfunctions of the derivative operator $\frac{d}{dx}$ (whose generalised eigenfunctions are then their derivatives w.r.t. ξ of the form $x^m \mathrm{e}^{i\xi x}$, $m \in \mathbb{Z}_{>0}$) and they give rise naturally to an inner-product, $\langle \cdot \mid \cdot \rangle$, under which $\langle \mathrm{e}^{i\xi x} \mid \mathrm{e}^{i\rho x} \rangle = \delta_0(\xi - \rho)$; with the associated Fourier transform being $\mathcal{F}[f](\xi) = \langle f(x) \mid \mathrm{e}^{i\xi x} \rangle = \int_{-\infty}^{\infty} f(x) \mathrm{e}^{-i\xi x} \, \mathrm{d}x$.

In the generalised Césaro framework the central constituents are functions which are pure powers of x, namely x^s, $s \in \mathbb{C}$. These are the eigenfunctions of the Césaro operator, P, and their derivatives w.r.t. s, namely $x^s (\ln x)^m$ ($m \in \mathbb{Z}_{>0}$) are likewise important as the generalised eigenfunctions of P.

In analogy with Fourier theory, it is reasonable to ask whether there might be a natural inner-product, $\langle \cdot \mid \cdot \rangle$, under which $\langle x^s \mid x^\rho \rangle = \delta_0(s - \rho)$; with an associated "Taylor transform", $\mathcal{T}$, given by $\mathcal{T}[f](s) = \langle f(x) \mid x^s \rangle$ and defined by some sort of integral whose integrand involves $f(x)$ and x^s.

Such an inner product and transform might in turn be used to pick out the coefficients of powers, x^m ($m \in \mathbb{Z}$), in the Taylor series (or other power series) expansion for a function, $f(x)$, near 0 (albeit we have seen in [XII] that we might simultaneously have to take into account the corresponding expansions near ∞ also).

Such speculation is both bolstered and refined by reviewing many of the examples we considered in [XI] and [XII], and the observations and conjectures we developed from them. In particular, in [XII] we considered the example of $f(x) = \ln(1+x)$ and saw that its log-divergence as $x \to \infty$ corresponded to $\check{f}(s)$ having a simple-pole at $s = 0$; and we extended this to see that a divergence of the form $x^{s_0} (\ln x)^m$ in $f(x)$ (at either 0 or ∞ per conjecture 1 in [XII]) leads to a pole of order m in $\check{f}(s)$ at $s = s_0$.

If we were, say, to multiply $\check{f}(s)$ by $\frac{2\pi}{\sin(2\pi s)}$, then we would get a unified state of affairs under which coefficients of pure powers x^{s_0} correspond to the residues of simple poles of this resulting function in s, while each extra power of $\ln x$ attached to such a pure power then increases the order of the associated pole by 1.

This is very reminiscent of the behaviour of functions of x with an associated complex parameter, s, which we have observed within the generalised Césaro framework. We saw this all the way back in [I] in our demonstration by Césaro means that the simple pole at $s = 1$ in the analytic continuation of $\zeta(s)$ corresponds to the place where its p-sum function develops a logarithmic divergence; likewise in many other examples in [I]-[III] and subsequent papers; similarly in our discussion of the netting of poles arising separately from the contributions of NT_+ and NT_- among the non-trivial roots in the generalised root identities for ζ in [VIII]; and so on.

Since, moreover, any power, x^m or x^{s_0} generally, is always divergent classically either at 0 or ∞ - and would remain so under anti-differentiation as part of any inner-product or Taylor transform - all of these considerations suggest two conclusions.

First, that we should adopt a generalised Césaro perspective in trying to define such an inner-product and such a transform $\mathcal{T}$. And secondly that, once defined, we should expect such structures ($\langle \cdot \mid \cdot \rangle$ and $\mathcal{T}$) to create a straightforward connection between, on the one hand, the poles and residues of $\overset{\vee}{f}(s)$, or rather of $\overset{\vee}{f}(s) \cdot \frac{2\pi}{\sin(2\pi s)}$; and, on the other, the places (in s-space) where the integrals involved take on log-divergences as $x \to 0$ or $x \to \infty$.

With these considerations in mind, we now turn to finding a specific definition of such an inner-product and associated Taylor transform.

2.1 Definition of inner-product $\langle \cdot \mid \cdot \rangle$ and Taylor transform $\mathcal{T}$

Whatever $\langle \cdot \mid \cdot \rangle$ is, it should have $\langle x^{s_0} \mid x^s \rangle = 0$ whenever $s \neq s_0$, so that $\{x^s\}_{s \in \mathbb{C}}$ is a collection of orthogonal functions.

Now the anti-derivative of any integrand, x^ρ, evaluated at X is $\frac{X^{\rho+1}}{\rho+1}$, and in the generalised Césaro framework suggested by the above discussion, this converges to 0 unless $\rho = -1$, in which case the integrand was x^{-1} and we get a log-divergence with no generalised Césaro limit.

It thus makes sense to try defining $\mathcal{T}$ via a generalised Césaro integral on $[0, \infty)$, and one in which x^{s_0} and x^s combine to give an integrand x^ρ where $\rho \neq -1$ whenever $s \neq s_0$ and $\rho = -1$ when $s = s_0$. Given that the natural Haar measure on $[0, \infty)$ is $\frac{dx}{x}$, it is therefore natural to try defining

$$\mathcal{T}[x^{s_0}](s) \;=\; \langle x^{s_0} \mid x^s \rangle \;:=\; \int_0^\infty x^{s_0 - s - 1}\, dx \;=\; \int_0^\infty x^{s_0 - s}\, \frac{dx}{x} \quad . \qquad (1)$$

Moving from single powers to finite sums of such powers and then to arbitrary functions, this would suggest we finally define the inner-product and Taylor-transform of arbitrary functions $f(x)$ and $g(x)$ as the generalised Césaro integrals

$$\langle f(x) \mid g(x) \rangle \;:=\; \int_0^\infty f(x) \cdot g\left(\frac{1}{x}\right)\, \frac{dx}{x} \qquad (2)$$

and

$$\mathcal{T}[f](s) \;=\; \langle f(x) \mid x^s \rangle \;:=\; \int_0^\infty f(x) \cdot x^{-s} \, \frac{dx}{x} \;=\; \int_0^\infty x^{-s-1} \cdot f(x) \, dx \quad . \tag{3}$$

Comment: In the above, we assume for simplicity for the moment that f and g are non-singular on $(0, \infty)$. Thus, the integrals in the above definitions being generalised Césaro integrals means that $\int_0^\infty$ is to be interpreted as $\underset{X,Y\to\infty}{Clim} \int_{\frac{1}{Y}}^{X}$. As discussed in [I]-[III] we may occasionally simplify this further as $\underset{X\to\infty}{Clim} \int_{\frac{1}{X}}^{X}$, but only where we are careful to ensure that this does not lead to improper cancellation among contributions near 0 and near ∞, which might introduce errors into the two separate Césaro limits.

Now, as a first observation, note that under this definition $\langle \cdot \mid \cdot \rangle$ is symmetric, meaning that

$$\langle f \mid g \rangle = \langle g \mid f \rangle \quad . \tag{4}$$

This follows immediately under the substitution $u = \frac{1}{x}$, since then $\frac{dx}{x} = -\frac{du}{u}$, so that, formally,

$$\langle f \mid g \rangle \;=\; \int_0^\infty f(x) \cdot g\left(\frac{1}{x}\right) \frac{dx}{x} \;=\; \int_0^\infty f\left(\frac{1}{u}\right) \cdot g(u) \, \frac{du}{u} \;=\; \langle g \mid f \rangle$$

and this continues to hold under a more rigorous application of the Césaro integral definition above, since the space spanned by the set of power and power-log functions which have generalised C'esaro limit zero is invariant under this substitution.

As a second observation, note also that the Taylor-transform $\mathcal{T}$ as defined in equation 3 is of course almost identical to the familiar Mellin transform. Specifically, we have formally that

$$\mathcal{T}[f](s) = \mathcal{M}[f](-s) \quad . \tag{5}$$

At first glance it might thus appear that we have merely rediscovered $\mathcal{M}$. But in fact we have done more than this, because our definition of $\mathcal{T}$ places it within the framework of generalised Césaro convergence, rather than purely the domain of classical convergence to which the theory of Mellin transforms has always thus far been restricted.

This is a significant extension. To begin with, it has already allowed us to express $\mathcal{T}$ (and thereby $\mathcal{M}$) in terms of a natural inner-product. This was impossible before, because no power or power-log functions are classically integrable on $[0, \infty)$. For example, the key orthogonality property that $\langle x^s \mid x^{s_0} \rangle := \int_0^\infty x^{s-s_0-1} \, \mathrm{d}x$ is zero whenever $s \neq s_0$ was ill-posed within the world of classical convergence, but is both true and trivial within the generalised Césaro setting.

Moreover, as we shall see in the rest of this section and in the remaining two sections of this paper, this extension not only increases the domain of applicability of $\mathcal{M}$ - allowing us to apply it to many divergent or singular functions to which it has not been hitherto applicable; it also ends up greatly simplifying the expression of many key properties of $\mathcal{M}$ and core aspects of its theory.

In light of all this, we shall generally work exclusively with $\mathcal{M}$ going forward and will, for the most part, avoid further reference to $\mathcal{T}$ in order to minimise duplication. In doing so, however, we take the definition of $\mathcal{M}$ via

$$\mathcal{M}[f](s) := \int_0^\infty f(x) \cdot x^{s-1}\,\mathrm{d}x$$

as now being a generalised Césaro integral definition in all that follows.

In short, we embrace the idea that the generalised Césaro convergence framework is, in fact, the right setting for the definition and analysis of the Mellin transform, and we take this as its setting in everything that follows, including the promised demonstrations of the way in which this simplifies much of its theory and key properties.

2.2 Generalised Césaro analysis and the poles and residues of $\mathcal{M}[f](s)$

With this stipulation, let us now make explicit how $\langle\cdot\mid\cdot\rangle$ and $\mathcal{M}$ create a direct connection between the poles and residues of $\overset{\vee}{f}(s)\cdot\frac{2\pi}{\sin(2\pi s)}$ and the locations in s-space where, under our generalised Césaro paradigm, the integral defining $\mathcal{M}[f](s)$ has log-divergences.

Based on the heuristic results of [XI] and [XII], recall first that the TLA-coefficient function, $\overset{\vee}{f}(s)$, is related to the Mellin transform of f by

$$\overset{\vee}{f}(s) = -\mathcal{M}[f](-s)\cdot\frac{\sin(2\pi s)}{2\pi} \quad . \tag{6}$$

In [XI] this was derived by Taylor-series-to-the-left methods of integration, but now that we have established $\mathcal{M}$ within its generalised Césaro domain, we may once again take this as *defining* the canonical form of $\overset{\vee}{f}(s)$.

In [XII] we further saw in many examples that, for $s = m \in \mathbb{Z}$, $\overset{\vee}{f}(s)$ is given by

$$\overset{\vee}{f}(m) = f_0(m) - f_\infty(m) \tag{7}$$

where $f_0(m)$ is the coefficient of x^m in the power series for $f(x)$ for x near 0, and $f_\infty(m)$ is the coefficient of x^m in the power series for $f(x)$ as $x\to\infty$.

Now suppose for simplicity we initially consider a Césaro-integrable function, $f(x)$, which is Schwartzian as $x\to\infty$, and suppose further that $f(x)$ has power series expression $\sum_{j=m_0}^{\infty} a_j x^j$ for x near 0 (note that m_0 may be negative). Then in equation 7 we have

$$\overset{\vee}{f}(m) = f_0(m) = a_m \quad \text{for any} \quad m\in\mathbb{Z}\,.$$

It would follow at once from re-expressing equation 6 as

$$\mathcal{M}[f](-s) = -\overset{\vee}{f}(s) \cdot \frac{2\pi}{\sin(2\pi s)}$$

that for $s = m+\epsilon$, ϵ small, we have $\mathcal{M}[f](-m-\epsilon) = -(a_m+O(\epsilon))\cdot\left(\frac{2\pi}{2\pi\epsilon} + O(1)\right) \approx -\frac{a_m}{\epsilon}$. Thus, based on our heuristic results from [XI] and [XII], we would expect that $\mathcal{M}[f](s)$ has a simple pole at $s = -m$ with residue a_m.

Let us now see how this can alternatively be deduced directly from the generalised Césaro definition of $\mathcal{M}$.

Since f is Césaro-integrable and Schwartzian as $x \to \infty$, $\int_a^\infty f(x) \cdot x^{-s-1}\,\mathrm{d}x$ is locally uniformly bounded for any $a \in \mathbb{R}_{>0}$. We are focused on singular behaviour in ϵ and so it follows that, in the Césaro definition of $\mathcal{M}[f](-s)$ for $s = m + \epsilon$ near m, we can focus exclusively on $\int_{\frac{1}{Y}} f(x) \cdot x^{-s-1}\,\mathrm{d}x$. Here, by omitting the upper limit in this notation we are indicating that we are neglecting evaluation at any upper-limit (since this just affects the constant term in the resulting expression in ϵ) and concentrating exclusively on the behaviour, in the generalised Césaro limit as $Y \to \infty$, arising from evaluation at the lower limit.

Now, we divide the power series for $f(x)$ near 0 into three components as

$$\sum_{j=m_0}^{\infty} a_j x^j = \left\{ \sum_{j=m_0}^{m-1} a_j x^j \right\} + a_m x^m + \left\{ \sum_{j=m+1}^{\infty} a_j x^j \right\} \quad . \tag{8}$$

Focusing initially on the specific value $s = m$ in $\int_{\frac{1}{Y}} f(x) \cdot x^{-m-1}\,\mathrm{d}x$, the first component-set then gives us $\sum_{j=m_0}^{m-1} a_j \frac{Y^{m-j}}{m-j}$, which is a polynomial in Y with generalised Césaro limit 0 as $Y \to \infty$.

As for the third component-set, it gives us an asymptotic series $\sum_{j=m+1}^{\infty} a_j \frac{Y^{m-j}}{m-j}$ which converges classically to zero as $Y \to \infty$.

And the term $a_m x^m$ gives a logarithmic divergence in Y, namely $a_m \ln Y$, with no Césaro limit. Thus it follows that $\mathcal{M}[f](-s)$ is not well-defined at $s = m$ (as expected).

Turning to $s = m+\epsilon$, ϵ small, we have likewise that, in $\int_{\frac{1}{Y}} f(x)\cdot x^{-s-1}\,\mathrm{d}x$, the first component-set gives us $\sum_{j=m_0}^{m-1} a_j \frac{Y^{m+\epsilon-j}}{m+\epsilon-j}$. This is still a linear combination of powers of Y with generalised Césaro limit zero via a regular polynomial, $q_1(\epsilon; P)$, which is non-singular and all of whose roots lie strictly away from 1 for ϵ small.

And the third component-set still gives us an asymptotic series $\sum_{j=m+1}^{\infty} a_j \frac{Y^{m+\epsilon-j}}{m+\epsilon-j}$ converging classically to zero as $Y \to \infty$ without the need for application of any regular polynomial in P.

As for the term $a_m x^m$, it gives $a_m \frac{Y^\epsilon}{\epsilon}$. This converges in a generalised Césaro sense to zero for any $\epsilon \neq 0$, via the regular polynomial $q_2(\epsilon; P) = \left(\frac{1+\epsilon}{\epsilon}\right)\left(P - \frac{1}{1+\epsilon}\right)$, but $q_2(\epsilon; P)$ is singular and its root approaches 1 as $\epsilon \to 0$.

Thus, by our standard Césaro L'Hopital's calculation (see [I]), it follows that

$$\begin{aligned} \underset{\epsilon\to 0}{lim}\, \epsilon \cdot \mathcal{M}[f](-m-\epsilon) &= \underset{Y\to\infty}{Clim}\, \underset{\epsilon\to 0}{lim}\, ((1+\epsilon)P-1)\left[\int_{\frac{1}{Y}}^{\infty} f(x)\cdot x^{-m-1-\epsilon}\,\mathrm{d}x\right] \\ &= a_m \cdot \underset{Y\to\infty}{Clim}\,(P-1)[\ln Y] = -a_m \quad . \end{aligned}$$

We indeed thus confirm by direct calculation that $\mathcal{M}[f](-m-\epsilon) \approx -\frac{a_m}{\epsilon}$ so that $\mathcal{M}[f](s)$ has a simple pole at $s=-m$ with residue a_m.

Overall we see that, as promised, the generalised Césaro definition of $\mathcal{M}[f](s)$ yields a pole, with residue a_m, at the point $s=-m$ where the integrand in the defining integral $\int_0^\infty f(x)\cdot x^{s-1}\,\mathrm{d}x$ acquires a term $a_m \cdot \frac{1}{x}$ and hence where the associated Césaro defining integral $\int_{\frac{1}{Y}}^{X} f(x)\cdot x^{s-1}\,\mathrm{d}x$ attracts a pure log-divergence.

In turn, it is then easy to see, by differentiating w.r.t. s, that a pole of order N at $s=-m$ corresponds to a place where the integrand in $\mathcal{M}[f](s)$ contains a term of the form $x^m(\ln x)^{N-1}$.

Comments: (a) The sorts of results that we have just proved - that terms of the form $a_m x^m$ in an asymptotic power series for $f(x)$ near 0 correspond to simple poles with residue a_m at $s=-m$ in the Mellin transform $\mathcal{M}[f](s)$; and that this generalises to a correspondence between terms of the form $a_m x^m(\ln x)^{N-1}$ and poles at $s=-m$ of order N - are not new in the theory of the Mellin transform.

But such results in the existing theory of $\mathcal{M}$ often require many more technical conditions - on the permissible nature of the function f; growth restrictions on it dependent on m; restrictions on permissible s-values etc - than we have needed. So it is worth noting precisely how the above argument and its placement within a generalised Césaro framework, extends such existing results and simplifies their formulation.

First, unlike in traditional such results, the function f may be singular as $x\to 0$, with a pole of finite order (if $m_0<0$) there, since our generalised Césaro definition of $\mathcal{M}[f](s)$ handles such a singularity seamlessly, without convergence difficulties.

Secondly, the very stringent condition we imposed that $f(x)$ be Schwartzian as $x\to\infty$, was not necessary and was only imposed for simplicity and clarity of argument. We could even have $f(x)$ divergent, with a power series as $x\to\infty$ of the form $\sum_{j=-\infty}^{m_\infty} \tilde{a}_j x^j$ for some $m_\infty \in \mathbb{Z}$, and the same result would hold as long as $\tilde{a}_m = 0$ (i.e. no x^m term in this power series as $x\to\infty$) by a minor extension of the same generalised Césaro reasoning.

Indeed, in line with our heuristic results in equations 6 and 7, even if $\tilde{a}_m \neq 0$, then a similar result would hold, just with $\mathcal{M}[f](s)$ having a simple pole at $s=-m$ with residue $a_m - \tilde{a}_m$, and we will prove this in the next section.

Thus, not only do we not need to impose growth conditions on f as $x\to\infty$ but, unlike in results of this sort in the existing theory of the Mellin transform, we can avoid having to focus exclusively on 0 or ∞ *separately* and instead deduce results that *simultaneously* take account of the behaviour of f near both 0 and ∞.

And finally, the argument above within the generalised Césaro context does not need to impose any conditions on the domain of s-values where it is applicable, or engage in technical restriction of this s-domain to avoid classical non-integrability problems. Instead it works straightforwardly on unrestricted $s \in \mathbb{C}$ without caveat. We shall return to the way in which the generalised Césaro framework for defining $\mathcal{M}$ leads to major simplification in the expression of results for Mellin transforms, and avoids technicalities in regard to domains of applicable s-values in section 2.4.

(b) The result we have proved has been for $m \in \mathbb{Z}$. However, the same argument will apply if the power series for $f(x)$ near 0 contains a term ax^{s_0}, $s_0 \notin \mathbb{Z}$.

In this case we would deduce a simple pole with residue a in $\mathcal{M}[f](s)$ at $s = -s_0$; and if the term were instead $ax^{s_0}(\ln x)^{N-1}$ we would deduce there a pole of order N.

The only reasons we have focused on the case of $s_0 = m \in \mathbb{Z}$ above are, first, because power series in integer powers at 0 or ∞, without branch cuts, arise more commonly in functions of interest; and secondly, because in this case, for a term of the form $a_m x^m$, we recover $\overset{\vee}{f}(m)$ as having the value a_m from equation 6 (since the zero of $\frac{\sin(2\pi s)}{2\pi}$ at $s = -m$ cancels the simple pole there in $\mathcal{M}[f](-s)$). By contrast, $\overset{\vee}{f}(s_0)$ remains singular under equation 6 when $s_0 \notin \mathbb{Z}$.

In [XI] and [XII] the TLA-coefficient function, $\overset{\vee}{f}(s)$, was our primary object of interest and we saw in many examples that we can in fact often deduce $\overset{\vee}{f}(s)$ directly from consideration simply of $\overset{\vee}{f}(m)$, $m \in \mathbb{Z}$ - allowing us, for example, to immediately deduce $\mathcal{M}[f](-s)$ via equation 6, or to conduct local-to-global inference, or to perform integration on $[0, \infty)$ etc. Thus the case of $m \in \mathbb{Z}$ is of special importance also in the world of Taylor-series-to-the-left methods developed in [XI] and [XII].

2.3 An integrated perspective

In our discussion in the last section of the poles and residues of $\mathcal{M}[f](s)$ under a generalised Césaro framework, we saw how the rigorous calculation of $\mathcal{M}[f](s)$ confirmed the heuristic results and relationships, such as equations 6 and 7, which we had derived in [XI] and [XII] based on Taylor-series-to-the-left methods.

In fact, the adoption of the generalised Césaro framework for $\mathcal{M}$, and the way it then allows us to take $-\mathcal{M}[f](-s) \cdot \frac{\sin(2\pi s)}{2\pi}$ as the canonical *definition* of the TLA-coefficient function $\overset{\vee}{f}(s)$, provides a unified framework under which all of the concepts and intuitive argumentation of [XI] and [XII] combine with the extended Césaro definition of the Mellin transform to provide an integrated whole - a set of ideas, relationships and results which are now all rigorous and interconnected, rather than merely being "morally right" as heuristic conjectures.

For example, conjecture 1 from [XI] asserted that, for f smooth on $[0,\infty)$ and sufficiently rapidly decaying as $x \to \infty$ as to ensure classical integrability, we have $\overset{\vee}{f}(-1) = 0$ and

$$\int_0^\infty f(x)\,\mathrm{d}x = -\overset{\vee}{f}'(-1) = -\lim_{\epsilon \to 0} \frac{\overset{\vee}{f}(-1+\epsilon)}{\epsilon} \quad .$$

This now follows immediately from the canonical definition of $\overset{\vee}{f}(s)$ as

$$\overset{\vee}{f}(s) = \mathcal{M}[f](-s) \cdot \frac{\sin(2\pi s)}{2\pi} \quad .$$

For the definition of $\mathcal{M}[f](s)$ as the generalised Césaro integral $\int_0^\infty x^{s-1} \cdot f(x)\,\mathrm{d}x$ means that $\mathcal{M}[f](1) = \int_0^\infty f(x)\,\mathrm{d}x$ and since this is finite (by classical integrability), so $\overset{\vee}{f}(-1) = -\mathcal{M}[f](1) \cdot \frac{\sin(-2\pi)}{2\pi} = 0$. But then

$$\begin{aligned}
\overset{\vee}{f}'(-1) &= \frac{d}{ds}\left(-\mathcal{M}[f](-s) \cdot \frac{\sin(2\pi s)}{2\pi}\right)\Bigg|_{s=-1} \\
&= \left\{-\mathcal{M}[f](-s) \cdot \cos(2\pi s) + \frac{d}{ds}\mathcal{M}[f](-s) \cdot \frac{\sin(2\pi s)}{2\pi}\right\}\Bigg|_{s=-1} \\
&= -\mathcal{M}[f](1) \cdot 1 + 0 \;=\; -\int_0^\infty f(x)\,\mathrm{d}x
\end{aligned}$$

and so the result follows immediately.

Here f must decay sufficiently rapidly to ensure that $\frac{d}{ds}\mathcal{M}[f](-s) \mid_{s=-1} = \int_0^\infty f(x) \ln x\,\mathrm{d}x$ is also finite (i.e. $f(x)\ln x$ is also classically integrable) and this should be included in the required conditions.

But, subject to this stipulation, the key point is that conjecture 1 from [XI] does now become a rigorous result rather than a heuristic claim. And indeed, since the Mellin transform is now located within a generalised Césaro framework, the proof above applies equally well to prove the same result much more broadly - for any function, f, which is Césaro-integrable on $[0,\infty)$ and whose power series near 0 and near ∞ contain no term of the form $a\frac{(\ln x)^{N-1}}{x}$, so that $f(x)\ln x$ is also Césaro-integrable on $[0,\infty)$.

Conjectures 2 and 3 from [XI] similarly follow in trivial fashion under our canonical definition of $\overset{\vee}{f}(s)$ as $-\mathcal{M}[f](-s) \cdot \frac{\sin(2\pi s)}{2\pi}$ and they likewise become more general results under our generalised Césaro framework for the Mellin transform - results applicable not just to smooth and rapidly-decaying functions, but to potentially divergent and classically non-integrable functions as long as they are suitably Césaro-integrable.

In turn the rigourisation of these three conjectures affirms the validity and usefulness of all the Taylor-series-to-the-left methods by which these results were first derived in [XI]. These include lemmas 1a-1c in [XI] which facilitate the derivation of the canonical form of $\overset{\vee}{f}(s)$ directly without having to calculate separately the Mellin transform of f (indeed they facilitate the indirect calculation of the Mellin transform from them); their application in bootstrapping methods of integration; and so on.

And in the same fashion we can now, as promised, prove the main claim developed in [XII]. This is the result encapsulated in equation 7 here and stated as conjecture 1 in [XII], namely:

Result: *Suppose f is a smooth function on $(0, \infty)$ and suppose that for x near 0, f has the power series expansion $\sum_{j=m_0}^{\infty} f_0(j)x^j$ for some integer m_0; and that as $x \to \infty$, f has the power series expansion $\sum_{j=-\infty}^{m_\infty} f_\infty(j)x^j$ for some integer m_∞. Here, the power series of which $\{f_0(j)\}_{j=m_0}^{\infty}$* and *$\{f_\infty(j)\}_{j=-\infty}^{m_0}$ are the coefficient sequences may be Taylor or Laurent series with some finite or infinite radius of convergence, R; or may merely be asymptotic expansions not convergent for any x* (so $R = 0$). *Then if f is integrable in a generalised Césaro sense on $[0, \infty)$ it has a canonical Taylor-coefficient function, $\overset{\vee}{f}(s)$ $(s \in \mathbb{C})$, and this satisfies that*

$$\overset{\vee}{f}(m) = f_0(m) - f_\infty(m) \quad \text{for all} \quad s = m \in \mathbb{Z}\,. \tag{9}$$

Proof: The argument follows the same lines as outlined in previous derivations, but we formalise and tighten it.

Write $f(x)$ as $f(x) = g(x) + h(x)$ where $g(x) = f(x) \cdot u_{[0,1]}(x)$ and $h(x) = f(x) \cdot u_{(1,\infty)}(x)$; and where the function $u_I(x)$ on an interval I has value 1 when $x \in I$ and 0 otherwise. Then $f(x) = g(x) + h(x)$ and $\overset{\vee}{f}(s) = \overset{\vee}{g}(s) + \overset{\vee}{h}(s)$. Now fix $m \in \mathbb{Z}$ arbitrary and consider $s = m + \epsilon$ near m.

Considering $g(x)$ first, we have $-\mathcal{M}[g](-s) = -\underset{Y\to\infty}{\mathcal{C}lim} \int_{\frac{1}{Y}}^{1} g(x) \cdot x^{-s-1}\, \mathrm{d}x$. Since we multiply $-\mathcal{M}[g](-s)$ by $\frac{\sin(2\pi s)}{2\pi}$ to get $\overset{\vee}{g}(s)$, and since $\sin(2\pi s)$ has a zero of order 1 at $s = m$, we need only focus on any singular behaviour of order 1 in $-\mathcal{M}[g](-s)$ and ignore any uniformly bounded finite contributions to $-\mathcal{M}[g](-s)$ as s approaches m.

Now, writing $f_0(j) = a_j$ and dividing the power series for $g(x)$ near $x = 0$ into three pieces as before, we have

$$\sum_{j=m_0}^{\infty} a_j x^j = \left\{ \sum_{j=m_0}^{m-1} a_j x^j \right\} + a_m x^m + \left\{ \sum_{j=m+1}^{\infty} a_j x^j \right\}$$

and then in $-\mathcal{M}[g](-s)$ the first component-set gives us

$$-\underset{Y\to\infty}{Clim}\left\{\sum_{j=m_0}^{m-1}\int_{\frac{1}{Y}}^{1} a_j x^{j-s-1}\,\mathrm{d}x\right\} = \underset{Y\to\infty}{Clim}\left\{\sum_{j=m_0}^{m-1} a_j \frac{Y^{s-j}}{s-j}\right\} + C_1(s) = C_1(s)$$

where $C_1(s)$ is uniformly bounded for s in a neighbourhood of m and the Césaro-limit is achieved via a regular polynomial $q_1(s;P)$ which is non-singular and all of whose roots lie strictly away from 1 in such a neighbourhood.

Similarly, the third component-set gives us

$$\underset{Y\to\infty}{Clim}\left\{\sum_{j=m+1}^{\infty} a_j \frac{Y^{s-j}}{s-j}\right\} + C_3(s) = C_3(s)$$

where $C_3(s)$ is uniformly bounded for s in a neighbourhood of m and the Césaro-limit is in fact a classical limit requiring no application of any regular polynomial in P.

As for the remaining term $a_m x^m$ this gives us $a_m \frac{Y^{s-m}}{s-m} + \frac{a_m}{s-m}$ which has generalised Césaro-limit $\frac{a_m}{s-m}$ via the regular polynomial $q_2(s;P) = \left(\frac{s-m+1}{s-m}\right)\left(P - \frac{1}{s-m+1}\right)$.

It follows finally that the singular part of $-\mathcal{M}[g](-s)$ is $\frac{a_m}{s-m}$ and thus $\overset{\vee}{g}(m) = a_m$ after multiplying by $\frac{\sin(2\pi s)}{2\pi}$ and taking the limit as $s \to m$.

In exactly the same way, on writing $f_\infty(j) = \tilde{a}_j$ and splitting the power series for $h(x)$ as $x \to \infty$ into three component-sets as

$$\sum_{j=-\infty}^{m_\infty} \tilde{a}_j x^j = \left\{\sum_{j=-\infty}^{m-1} \tilde{a}_j x^j\right\} + \tilde{a}_m x^m + \left\{\sum_{j=m+1}^{m_\infty} \tilde{a}_j x^j\right\}$$

we get in $-\mathcal{M}[h](-s) = -\underset{X\to\infty}{Clim}\int_1^X h(x)\cdot x^{-s-1}\,\mathrm{d}x$ that the first component-set gives us

$$\underset{X\to\infty}{Clim}\left\{\sum_{j=-\infty}^{m-1} \tilde{a}_j \frac{X^{j-s}}{s-j}\right\} + \tilde{C}_1(s) = \tilde{C}_1(s)$$

where $\tilde{C}_1(s)$ is uniformly bounded for s in a neighbourhood of m and the Césaro-limit is a classical limit requiring no application of a regular polynomial in P. And the third component-set gives us

$$\underset{X\to\infty}{Clim}\left\{\sum_{j=m+1}^{m_\infty} \tilde{a}_j \frac{X^{j-s}}{s-j}\right\} + \tilde{C}_3(s) = \tilde{C}_3(s)$$

where $\tilde{C}_3(s)$ is uniformly bounded for s in a neighbourhood of m and the Césaro-limit is achieved via a regular polynomial $\tilde{q}_3(s;P)$ which is non-singular and all of whose roots lie strictly away from 1 in such a neighbourhood.

And lastly, the term $\tilde{a}_m x^m$ gives us $a_m \frac{X^{m-s}}{s-m} - \frac{\tilde{a}_m}{s-m}$ which has generalised Césaro-limit $\frac{\tilde{a}_m}{s-m}$ via the regular polynomial $\tilde{q}_2(s;P) = \left(\frac{m-s+1}{m-s}\right)\left(P - \frac{1}{m-s+1}\right)$.

It follows finally that the singular part of $-\mathcal{M}[h](-s)$ is $-\frac{\tilde{a}_m}{s-m}$ and thus $\overset{\vee}{h}(m) = -\tilde{a}_m$ after multiplying by $\frac{\sin(2\pi s)}{2\pi}$ and taking the limit as $s \to m$.

Combining the contributions from $\overset{\vee}{g}(m)$ and $\overset{\vee}{h}(m)$ it follows finally that $\overset{\vee}{f}(m) = a_m - \tilde{a}_m$ and this completes the proof of the result. QED

We have thus now proven the main heuristic conjecture of [XII] and this cements a further aspect of the integrated perspective we have now established.

It shows the way in which the generalised Césaro framework for $\mathcal{M}$ reinforces our Taylor-series-to-the-left methodology and confirms that the TLA-coefficient function, $\overset{\vee}{f}(s)$, simultaneously combines information regarding the power-series behaviour of f near 0 and near ∞; and thereby facilitates the phenomenon of local-to-global inference which we explored in a number of examples in [XII]. We devote the (very short) next paper in this series to a further elegant, non-trivial example of such inference.

2.4 A simpler, cleaner formulation of the theory of Mellin transforms

In the last subsection we have seen how all of the concepts of [XI] and [XII], when combined with the theory of the Mellin transform under the generalised Césaro convergence framework, form a unified, integrated whole - one in which the results of [XI] and [XII] are all rigorously established and in which we may seamlessly invoke any and all of such ideas as Taylor-series-to-the-left methodology for integration and bootstrapping; TLA-coefficient functions; gauge-choices and canonical form for $\overset{\vee}{f}(s)$; simultaneous power series behaviour near 0 and ∞; and the generalised Césaro definition of $\mathcal{M}$ wherever and whenever it is useful to do so.

One such application is in seeing how the theory of the Mellin transform itself becomes much cleaner under the generalised Césaro framework, and how many of the technical conditions which are familiar in the classical treatment of this theory may be dispensed with, or much more simply expressed, under this generalised Césaro paradigm. Let us illustrate this with a well-known example.

The Mellin transform of $f(x) = \mathrm{e}^{-x}$: Under classical convergence in the existing treatment of its Mellin transform, we have that

$$\mathcal{M}[\mathrm{e}^{-x}](s) = \int_0^\infty x^{s-1} \cdot \mathrm{e}^{-x} \mathrm{d}x = \Gamma(s) \quad , \quad Re(s) > 0$$

but this only holds on the restricted domain $Re(s) > 0$, since for $Re(s) \leq 0$ $x^{s-1} \cdot \mathrm{e}^{-x}$ is non-integrable at $x = 0$.

To extend to the left we must change to $f(x) = \mathrm{e}^{-x} - 1$ and we then have that

$$\mathcal{M}[\mathrm{e}^{-x} - 1](s) = \Gamma(s) \quad , \quad -1 < Re(s) < 0 \quad .$$

Thus it is now $\mathrm{e}^{-x} - 1$, rather than e^{-x}, which has Mellin transform $\Gamma(s)$, but this is only true on the restricted domain $-1 < Re(s) < 0$. This is because the subtraction of 1 means the integrand $x^{s-1} \cdot (\mathrm{e}^{-x} - 1)$ now only becomes integrable at $x = 0$ when $Re(s) > -1$, but means it is no longer integrable as $x \to \infty$ unless $Re(s) < 0$.

In the same way we have in turn that

$$\mathcal{M}[\mathrm{e}^{-x} - 1 + x](s) = \Gamma(s) \quad , \quad -2 < Re(s) < -1$$

and

$$\mathcal{M}[\mathrm{e}^{-x} - 1 + x - \frac{1}{2}x^2](s) = \Gamma(s) \quad , \quad -3 < Re(s) < -2$$

and so on - so that all of these different functions have the same classical Mellin transform, but all on different, disjoint s-domains according to how much of the Taylor series for e^{-x} near 0 we have subtracted away in each case.

By contrast, under the generalised Césaro framework, things become much clearer. To begin with we have that the Mellin transform of e^{-x} is $\Gamma(s)$ for all $s \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, with no restriction on s-domain.

This is trivial to see because for any $s_0 \notin \mathbb{Z}_{\leq 0}$ the generalised Césaro definition of $\mathcal{M}[f](s_0)$ via $\int_{\frac{1}{Y}}^{X} x^{s_0 - 1} \cdot \mathrm{e}^{-x} \, \mathrm{d}x$ leads at most to a finite linear combination of divergent powers of Y, and this has a generalised Césaro limit as $Y \to \infty$ under a regular polynomial, $q(s, P)$, which is non-singular in a neighbourhood of s_0 and all of whose roots lie strictly away from 1.

Thus $\mathcal{M}[\mathrm{e}^{-x}](s)$ is well-defined for all $s \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$. And for $-1 < Re(s) < 0$ we have that

$$\begin{aligned} \mathcal{M}[\mathrm{e}^{-x}](s) &= \mathcal{M}[(\mathrm{e}^{-x} - 1) + 1](s) = \mathcal{M}[(\mathrm{e}^{-x} - 1)](s) + \mathcal{M}[1](s) \\ &= \Gamma(s) + \underset{X,Y \to \infty}{Clim} \int_{\frac{1}{Y}}^{X} x^{s-1} \, \mathrm{d}x \\ &= \Gamma(s) + \underset{X,Y \to \infty}{Clim} \left\{ \frac{X^s}{s} - \frac{Y^{-s}}{s} \right\} = \Gamma(s) \; . \end{aligned}$$

In the same way we can extend to $-2 < Re(s) < -1$ and so on, stepping one strip to the left at a time.

Thus we see that all the non-integrability issues and associated s-domain technical restrictions which beset the traditional theory of the Mellin transform of e^{-x} under classical convergence, evaporate like dew beneath the morning sun under the generalised Césaro definition of $\mathcal{M}$; and it is clear that the same is true for $f(x)$ in general.

This is because the generalised Césaro framework automatically handles any power-divergences which arise at $x = 0$ in the definition of $\mathcal{M}[f](s)$ from the power series expansion of f near 0; and it likewise naturally handles any such power-divergences arising as $x \to \infty$ in $\mathcal{M}[f](s)$ if f is not Schwartzian at ∞ as e^{-x} was, and instead has an asymptotic power series there of the form $\sum_{j=-\infty}^{m_\infty} a_j x^j$.

Another couple of perspectives help to clarify this simplification and how it plays out in other aspects of the theory of Mellin transforms.

Perspective 1: First, as we have seen, the technicalities arise in the classical theory from the presence of terms of the form $a_m x^m$ in the power series for $f(x)$ either near 0 or near ∞. If near 0, this requires $Re(s) > -m$ to ensure classical integrability; if near ∞ it requires $Re(s) < -m$. If a term of the same order $a_m x^m$ exists in both the power series near 0 and near ∞, then no classical Mellin transform is therefore even possible.

But this presents no problem for a Mellin transform based on generalised Césaro convergence. If $Re(s) > -m$, then $a_m x^{m+s-1}$ is classically integrable at 0 and $\int^X a_m x^{m+s-1} \mathrm{d}x$ gives a term $a_m \frac{X^{m+s}}{m+s}$ which is classically divergent as $X \to \infty$ but has generalised Césaro limit 0 via $\left(\frac{m+s+1}{m+s}\right)\left(P - \frac{1}{m+s+1}\right)$. Handling any other of a finite collection of terms which generate power-divergences (either at 0 or ∞, not necessarily simultaneously) in the same way we are thus able to calculate the generalised Césaro Mellin transform $\mathcal{M}[f](s)$ for any $Re(s) > -m$.

In the same way, for $Re(s) < -m$, then $a_m x^{m+s-1}$ is classically integrable as $x \to \infty$ and $\int_{\frac{1}{Y}} a_m x^{m+s-1} \mathrm{d}x$ gives a term $-a_m \frac{Y^{-m-s}}{m+s}$ which is classically divergent as $Y \to \infty$ (reflecting classical non-integrability near 0) but which has generalised Césaro limit 0 via $\left(\frac{m+s-1}{m+s}\right)\left(P + \frac{1}{m+s-1}\right)$; so that once again we have no difficulty in calculating $\mathcal{M}[f](s)$ for any $Re(s) < -m$.

In short, having a term $a_m x^m$ present in the power series for $f(x)$ both near 0 and near ∞ presents no issue in the new generalised Césaro paradigm for $\mathcal{M}$. It merely means that $f_0(m) = f_\infty(m)$ so that, in equation 7, we have $\overset{\vee}{f}(m) = 0$. In light of the relationship that $\mathcal{M}[f](s) = \overset{\vee}{f}(-s) \cdot \frac{2\pi}{\sin(2\pi s)}$, this reflects the fact that as $s \to m$ we do get simple poles developing in $\mathcal{M}[f](s)$ simultaneously from the influence of the term $a_m x^m$, on the one hand as $x \to 0$, and on the other as $x \to \infty$ - but these poles have offsetting residues $\pm a_m$ and therefore cancel each other out.

If, by contrast, $a_m x^m$ exists in only one of the power-series for $f(x)$ near 0 or near ∞, then there is no cancellation and we end up with a simple pole in $\mathcal{M}[f](s)$ at $s = -m$ with residue $\pm a_m$ ($+a_m$ if the term appears in the power series near $x = 0$; $-a_m$ if it appears in the power series as $x \to \infty$). But either way, we have no difficulty under our generalised Césaro paradigm in extending to calculate $\mathcal{M}[f](s)$ for arbitrary $s \in \mathbb{C}$, rather than being confined to a half-plane by the requirement of classical integrability.

Perspective 2: Another way of expressing this is that the pure powers, x^m, and indeed x^{s_0} ($s_0 \in \mathbb{C}$) more generally, are not only eigenfunctions of the

Césaro operator P, they are also therefore in the kernel of $\mathcal{M}$ as an operator on functions. That is:

$$Span\left(\{x^{s_0}\}\right) \subset Ker(\mathcal{M}) \quad . \tag{10}$$

Thus any finite linear combination of pure powers is also in $Ker(\mathcal{M})$, and in particular any polynomial in x and $\frac{1}{x}$.

This explains cleanly why, under our new generalised Césaro interpretation, all the functions e^{-x}, $\mathrm{e}^{-x} - 1$, $\mathrm{e}^{-x} - 1 + x$, $\mathrm{e}^{-x} - 1 + x - \frac{1}{2!}x^2, \ldots$, $\mathrm{e}^{-x} - \sum j = 0^N (-1)^j \frac{x^j}{j!} \ldots$. now have the same Mellin transform, $\Gamma(s)$; and why the domain of applicability of their Mellin transforms is now in all cases $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, with none of the strip-wise domain variability that arose previously purely as an artefact of requiring classical, rather than generalised Césaro, integrability.

This can alternatively be expressed in terms of the inverse Mellin transform. The inverse Mellin transform is defined in the same way as traditionally, as

$$\mathcal{M}^{-1}[\phi](x) := \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \phi(s) \cdot x^{-s} \, \mathrm{d}s \tag{11}$$

via a contour integral up a vertical line, γ_c, with real part c in the complex plane. Bearing in mind, however, that an inverse like $\mathcal{M}^{-1}$ is only well-defined up to $Ker(\mathcal{M})$, how does this play out differently as c varies under the two different frameworks of classical convergence and generalised Césaro convergence?

Well, illustrating with the case of $\phi(s) = \Gamma(s)$, as c moves left along the real axis we initially have $\mathcal{M}^{-1}[\Gamma](x) = \mathrm{e}^{-x}$ when $Re(c) > 0$. Then, as c crosses the simple pole of residue 1 in $\Gamma(s)$ at $s = 0$, $\mathcal{M}^{-1}[\Gamma](x)$ picks up a contribution from this pole and so becomes $\mathrm{e}^{-x} - 1$. This extra term arises from considering the integral over γ_c as an integral over γ_{c+1} minus an integral over a loop running up γ_{c+1}, left across to the "top" of γ_c, down γ_c and back right across to the "bottom" of γ_{c+1}, with this loop contributing the residue of the pole of $\Gamma(s)$ at $s = 0$ times x^0.

In turn, when c crosses the pole of $\Gamma(s)$ at $s = -1$ with residue -1 we need to subtract a loop-contribution consisting of this residue times x^1 (since x^{-s} is x^1 at the pole), leaving $\mathcal{M}^{-1}[\Gamma](x) = \mathrm{e}^{-x} - 1 + x$. And in general, when c crosses the pole of $\Gamma(s)$ at $s = -j$ with residue $\frac{(-1)^j}{j!}$, we need to subtract a loop contribution consisting of this residue times x^N, so that we get $\mathcal{M}^{-1}[\Gamma](x) = \mathrm{e}^{-x} - \sum_{j=0}^{N} (-1)^j \frac{x^j}{j!}$ on the strip $-(N+1) < Re(s) < -N$.[1]

Overall, we see why it is that, under classical convergence, we need to take each of these functions, $\mathrm{e}^{-x} - \sum_{j=0}^{N} (-1)^j \frac{x^j}{j!}$, in turn as the inverse Melin transform of $\Gamma(s)$, and why each then has a strip-wise domain-restriction attached to the appliocability of its Mellin transform. By contrast, within our generalised Césaro paradigm we can state simply that

$$\mathcal{M}^{-1}[\Gamma](x) = \mathrm{e}^{-x} \mod Ker(\mathcal{M})$$

[1] In all of these loop-contribution calculations we do need to justify omitting any contribution from the short horizontal components of the loop at the "top" and "bottom". This follows, however, because (per e.g. [1, section 11.1]) it is well-known that $\Gamma(s)$ decays exponentially like e^{-t} for $s = \sigma \pm it$ when $t \to \infty$.

and the fact that this inversion is now only defined modulo $Ker(\mathcal{M})$, which includes $Span\left(\{x^{s_0}\}\right)$, explains the fact that we actually get the different functions, $\mathrm{e}^{-x} - \sum_{j=0}^{N}(-1)^j \frac{x^j}{j!}$, depending on which vertical strip in the s-plane we choose to locate our inversion-contour, γ_c.

3 Additional examples and core properties of $\mathcal{M}$

The move to a generalised Césaro framework has engineered a much-enlarged domain of applicability and a simpler, cleaner expression of results regarding the Mellin transform, as well as facilitating validation of all the Taylor-series-to-the-left methods, bootstrapping and integration techniques, and local-to-global results broached in [XI] and [XII].

In this final section we conclude with a string of example sets illustrating these unified ideas. These affirm a number of standard Mellin transform results, as well as deducing a number of non-standard or new ones. They also demonstrate how the core properties of $\mathcal{M}$ from the traditional theory of the Mellin transform continue to hold, and may in some cases be approached from fresh angles.

Example set (i): [Pure localised powers]: In line with our earlier working it is trivial to calculate that, for any $a \in \mathbb{C}$ and any $N \in \mathbb{Z}_{\geq 0}$, we have

$$\mathcal{M}[u_{[0,1]} \cdot x^a](s) = \frac{1}{s+a} \quad \text{and} \quad \mathcal{M}[u_{(1,\infty)} \cdot x^a](s) = -\frac{1}{s+a} \tag{12}$$

while

$$\mathcal{M}[u_{[0,1]} \cdot x^a (\ln x)^{N-1}](s) = \frac{(-1)^{N-1}(N-1)!}{(s+a)^N} \tag{13}$$

and

$$\mathcal{M}[u_{(1,\infty)} \cdot x^a (\ln x)^{N-1}](s) = \frac{(-1)^{N}(N-1)!}{(s+a)^N} \quad . \tag{14}$$

These equations affirm standard results from the traditional theory of $\mathcal{M}$, but with the simplification that, in our generalised Césaro framework, they now hold for all $s \in \mathbb{C} \setminus \{a\}$, not just for a half-plane with either $Re(s) > a$ or $Re(s) < a$.

Example set (ii): [$f(x) = \frac{1}{1+x}$ and extensions]: Next consider $f(x) = \frac{1}{1+x}$, whose Mellin transform is $\mathcal{M}[f](s) = \int_0^\infty \frac{1}{1+x} \cdot x^{s-1}\, \mathrm{d}x$. In [XII] we derived from the power series for f near 0 and near ∞ that $\overset{\vee}{f}(s) = \cos(\pi s)$ and it thus follows immediately that

$$\mathcal{M}[\frac{1}{1+x}](s) = \overset{\vee}{f}(-s) \cdot \frac{2\pi}{\sin(2\pi s)} = \frac{\pi}{\sin(\pi s)} \quad .$$

Under our generalised Césaro framework this holds for all $s \in \mathbb{C} \setminus \mathbb{Z}$, not just on the strip $0 < Re(s) < 1$ as it does for the classical theory of the Mellin transform.

The form of $\mathcal{M}[f](s)$ here is of course consistent with the fact that $\frac{1}{1+x}$ has a Taylor series at 0 consisting of non-negative powers of x, x^j, each with coefficient $a_j = (-1)^j$; and an asymptotic power series as $x \to \infty$ consisting of negative powers of x, x^{-j}, each with coefficient $a_{-j} = (-1)^{j-1}$. Hence $\mathcal{M}[f](s)$ is a function on $\mathbb{C} \setminus \mathbb{Z}$, with poles at each $j \in \mathbb{Z}$ having residue $(-1)^j$.

The derivation of $\mathcal{M}[f](s)$ here by working indirectly using $\overset{\vee}{f}(s)$ and our Taylor-series-to-the-left ideas from [XI] and [XII] is much simpler than direct calculation (especially if restricted to classical convergence). However, such direct calculation is also achievable within the generalised Césaro framework without undue difficulty, using the Taylor-series-to-the-left bootstrapping and integration techniques developed in [XI].

Starting with $s = \frac{p}{q} \in \mathbb{Q}$ in lowest terms and making the substitution $u = x^{\frac{1}{q}}$, we have

$$\mathcal{M}[f](s) = q \cdot \int_0^\infty \frac{1}{1+u^q} \cdot u^{p-1}\,\mathrm{d}u \quad .$$

Writing the integrand $\frac{u^{p-1}}{1+u^q}$ as $g(u)$, we have from lemma 1a in [XI] that

$$\overset{\vee}{g}(\tilde{s}) = \frac{1}{2q} \cdot \frac{\sin\left(2\pi(\tilde{s}-p+1)\right)}{\sin\left(\pi\frac{(\tilde{s}-p+1)}{q}\right)} \quad .$$

It follows immediately in result 1 from [XI][2] that we have $\overset{\vee}{g}(-1) = 0$ and

$$\begin{aligned}\int_0^\infty g(u)\,\mathrm{d}u &= -\overset{\vee}{g}'(-1) \;=\; -\lim_{\epsilon\to 0}\frac{\overset{\vee}{g}(-1+\epsilon)}{\epsilon} \\ &= -\lim_{\epsilon\to 0}\frac{1}{2q}\cdot\frac{2\pi\epsilon}{\sin\left(-\pi\frac{p}{q}\right)}\cdot\frac{1}{\epsilon} \;=\; \frac{1}{q}\cdot\frac{\pi}{\sin\left(\pi\frac{p}{q}\right)}\end{aligned}$$

and thus

$$\mathcal{M}[f](\frac{p}{q}) = \frac{\pi}{\sin\left(\pi\frac{p}{q}\right)} \quad .$$

Since this result depends only on the quantity $\frac{p}{q}$, not p or q independently, it is then easy to go from arbitrary $s = \frac{p}{q} \in \mathbb{Q}$ to arbitrary $s \in \mathbb{R}_{>0} \setminus \mathbb{Z}_{>0}$ by taking a limit of a rational sequence approaching s; and finally to arbitrary $s \in \mathbb{C} \setminus \mathbb{Z}$ by analytic continuation. This proves that $\mathcal{M}[f](s) = \frac{\pi}{\sin(\pi s)}$ in general.

The same sort of bootstrapping arguments work equally well for $f(x) = \frac{1}{1+x^n}$ ($n \in \mathbb{Z}_{>0}$) to show that

$$\mathcal{M}[\frac{1}{1+x^n}](s) = \frac{\pi}{n} \cdot \frac{1}{\sin(\frac{\pi s}{n})}$$

[2] This was of course called conjecture 1 in [XI] but has now been proven

for all $s \in \mathbb{C} \setminus \{n\mathbb{Z}\}$; and more generally, for arbitrary $\nu \in \mathbb{R} \setminus \{0\}$ we have

$$\mathcal{M}[\frac{1}{1+x^{\nu}}](s) = \frac{\pi}{\mid \nu \mid} \cdot \frac{1}{\sin(\frac{\pi s}{\nu})} \quad . \tag{15}$$

Example set (iii) [Core properties and commutation]: The last result, equation 15, is in fact a special case of a general result that for arbitrary Césaro-integrable function, f, and $\nu \in \mathbb{R} \setminus \{0\}$, we have

$$\mathcal{M}[f(x^{\nu})](s) = \frac{1}{\mid \nu \mid} \cdot \mathcal{M}[f(x)](\frac{s}{\nu}) \tag{16}$$

so that scaling ($x \to x^{\nu}$) on the x-side corresponds under $\mathcal{M}$ to reciprocal dilation on the s-side ($s \to \frac{s}{\nu}$), together with an overall factor of $\frac{1}{|\nu|}$.

This is a well-known property of $\mathcal{M}$ from its classical theory and the point is simply to note that it - and all the other standard properties of $\mathcal{M}$ - continue to hold under our extension of the definition of $\mathcal{M}$ to the generalised Césaro framework; just on a larger class of Césaro-integrable (rather than merely classically-integrable) functions and without strip-wise domain restrictions.

We shall not recapitulate all such core properties of $\mathcal{M}$, precisely since they all continue to hold unaltered on these enlarged domains (of both applicable functions and permissible s-values) under the generalised Césaro paradigm. Tables of such core properties can readily be found in any reference on Mellin transform theory.

We merely note three things here:

(a) First, we shall return shortly to discuss the multiplicative and convolution properties among these in more detail in example set (v).

(b) Secondly, note that among these core properties are the following pair, which show that $\mathcal{M}$ is particularly well-behaved under the action of the operators $x \circ \frac{d}{dx}$ and $\frac{d}{dx} \circ x$:

$$\mathcal{M}[(x \circ \frac{d}{dx})^n f(x)](s) = (-s)^n \cdot \mathcal{M}[f(x)](s) \tag{17}$$

and

$$\mathcal{M}[(\frac{d}{dx} \circ x)^n f(x)](s) = (1-s)^n \cdot \mathcal{M}[f(x)](s) \quad . \tag{18}$$

There is a corresponding result for the action solely of $\frac{d}{dx}$, namely that

$$\mathcal{M}[(\frac{d}{dx})^n f(x)](s) = (-1)^n \frac{\Gamma(s)}{\Gamma(s-n)} \cdot \mathcal{M}[f(x)](s-n) \tag{19}$$

but it is less simple than either equation 17 or equation 18 and its extension from the world of classical convergence to the generalised Césaro world is less straightforward.

This should not, however, surprise us. This is because the essential message of this paper is that it is the generalised Césaro convergence framework which is the natural habitat for the definition of $\mathcal{M}$ and (per [II]) the operator $\frac{d}{dx} \circ x$ is in fact precisely the inverse of the Césaro operator, P, i.e.

$$P^{-1} = \frac{d}{dx} \circ x \quad . \tag{20}$$

Since $\frac{d}{dx} \circ x = x \circ \frac{d}{dx} + 1$ we also have that $x \circ \frac{d}{dx} = P^{-1} - 1$. Thus both the operators in equations 17 and 18 are closely related to P and certainly commute with it, whereas the operator $\frac{d}{dx}$ does not do so directly. This is what leads to the greater messiness of equation 19.

(c) The same sort of operator-commutation considerations lie behind the simplicity of the core properties regarding the behaviour of $\mathcal{M}$ under the action of dilation. For example we have simply that

$$\mathcal{M}[f(\nu x)](s) = \nu^{-s} \cdot \mathcal{M}[f(x)](s) \tag{21}$$

and this reflects the fact that dilation-invariance is a fundamental property of generalised Césaro convergence.

Likewise, the simplicity of equation 16 reflects the corresponding scaling-invariance of generalised Césaro convergence (see [II]).

Example set (iv): [A key function from the theory of ζ]: Applying property 16 with $\nu = n \in \mathbb{Z}_{>1}$ to the function e^{-x} considered earlier in this paper, gives us that

$$\mathcal{M}[\mathrm{e}^{-x^n}](s) = \frac{1}{n} \cdot \Gamma\left(\frac{s}{n}\right)$$

with the poles on the RHS now spread out to occur only on the lattice $n\mathbb{Z}_{\leq 0}$, reflecting the wider n-fold spacing in the terms in the Taylor series for $\mathrm{e}^{-x^n} = 1 - x^n + \frac{1}{2!}x^{2n} - \ldots$ near 0 (and with residues at these poles also suitably adjusted).

In particular, for $n = 2$ we get that

$$\mathcal{M}[\mathrm{e}^{-x^2}](s) = \frac{1}{2} \cdot \Gamma\left(\frac{s}{2}\right) \quad .$$

Taking $\nu = \sqrt{\pi}j$ and combining this with property 21, it in turn follows that

$$\mathcal{M}[\mathrm{e}^{-\pi j^2 x^2}](s) = \frac{1}{2}\pi^{-\frac{s}{2}} j^{-s} \cdot \Gamma\left(\frac{s}{2}\right) \quad . \tag{22}$$

Now recall the function $H(x) := \frac{1}{2} + \sum_{j=1}^{\infty} \mathrm{e}^{-\pi j^2 x^2}$ which we considered at length in [IV]-[VI] and which plays a crucial role in many key results regarding the Riemann zeta function, ζ. It follows from equation 22 that we have the following elegant relationship:

$$\mathcal{M}[H(x)](s) = \frac{1}{2}\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) \quad , \quad s \in \mathbb{C} \setminus \{0, 1\} \quad . \tag{23}$$

Here, recall that as we showed in [IV], $H(x)$ satisfies

$$H(x) = \frac{1}{2} + \mathcal{S}_\infty(x) \quad \text{as } x \to \infty \qquad \text{and} \qquad H(x) = \frac{1}{2}\frac{1}{x} + \mathcal{S}_0(x) \quad \text{as } x \to 0 \ . \tag{24}$$

Thus the clean expression of the beautiful Mellin transform result for $H(x)$ which we have obtained in equation 23 *requires* that we be operating within the generalised Césaro framework we have adopted (as does our indifference to whether $Re(s) > 1$ or not in claiming that $\sum_{j=1}^{\infty} j^{-s} = \zeta(s)$).[3]

Equation 23 also captures immediately the detail of the power series relationships for $H(x)$ given in equation 24.

These mean that $\mathcal{M}[H(x)](s)$ should have only two poles - one at $s = 0$ with residue $-\frac{1}{2}$ and one at $s = 1$ with residue $\frac{1}{2}$. And this is precisely what we get from $\frac{1}{2}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$ on the RHS in equation 23 since, on the one hand $\zeta(0) = -\frac{1}{2}$ and $\Gamma(\frac{s}{2}) \approx 2 \cdot \frac{1}{s}$ as $s \to 0$; and on the other $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $\zeta(s) \approx \frac{1}{s-1}$ for s near 1.

In traditional calculations involving $H(x)$ and many of the deep results regarding ζ which involve it[4], the need to avoid classical divergences and non-integrability often leads to working not with $H(x)$ itself, but with $G(x) := (x \circ \frac{d}{dx})(\frac{d}{dx} \circ x)H(x)$ instead.

The action of $(x \circ \frac{d}{dx})(\frac{d}{dx} \circ x)$ removes the constant term of $\frac{1}{2}$ in the behaviour as $x \to \infty$ and also removes the $\frac{1}{2}\frac{1}{x}$ divergence as $x \to 0$, so that $G(x)$ is now Schwartzian both as $x \to 0$ and as $x \to \infty$. This is reflected in the fact that now, by properties 17 and 18 applied to result 23, we have that

$$\mathcal{M}[G(x)](s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) \quad . \tag{25}$$

The $s(s-1)$ factors cancel the two poles that previously existed in $\mathcal{M}[H(x)](s)$ and leave the function on the RHS in equation 25 being an entire function, with no poles and with zeros which coincide with the nontrivial zeros of ζ on the critical strip.

Example set (v): [Using Taylor-series-to-the-left methods and TLA-coefficient functions in new ways for calculation of Mellin transforms]: Using the Taylor-series-to-the-left methods developed in [XI] and [XII] and illustrated in example set (ii), we can readily calculate Mellin transforms for many other well-known functions and then observe how the structure of their poles and residues reflects the exponents and coefficients of powers arising in the power series for each function near 0 and near ∞.

For example, using our formulae from [XI] for the TLA-coefficient functions of $f(x) = \cos x$ and $f(x) = \sin x$, together with the fact that $\mathcal{M}[f](s) = \overset{\vee}{f}(-s) \cdot$

[3] If we were operating under classical convergence, the Mellin transform would only be well-defined for $0 < Re(s) < 1$, but the summation of $\sum_{j=1}^{\infty} j^{-s}$ as $\zeta(s)$ would require $Re(s) > 1$.

[4] see e.g. [1, Chapter 11] giving Hardy's proof that there are infinitely many non-trivial zeros of ζ on the critical line.

$\frac{2\pi}{\sin(2\pi s)}$, we can immediately deduce that

$$\mathcal{M}[\sin x](s) = \sin(\frac{\pi s}{2}) \cdot \Gamma(s) \qquad \text{and} \qquad \mathcal{M}[\cos x](s) = \cos(\frac{\pi s}{2}) \cdot \Gamma(s)$$

with the replicating factors of $\sin(\frac{\pi s}{2})$ and $\cos(\frac{\pi s}{2})$ in each case acting to cancel off half the poles of $\Gamma(s)$ and leave only those at odd negative integers or even non-positive integers respectively. This derivation is much simpler than via traditional methods and also applies for general $s \in \mathbb{C}$, avoiding the s-strip restrictions that apply to these Mellin transforms as existing results under classical convergence.

We could continue in the same fashion, to obtain Mellin transforms of many other functions. However, rather than merely continue extending such a list of Mellin transform calculations using existing Taylor-series-to-the-left methods, we conclude this section and this paper with a discussion of one area where we believe this Taylor-series-to-the-left perspective opens up a further new way of approaching such calculations.

This is in the area of handling products of functions and associated Mellin convolution formulae.

A Taylor-series-to-the-left approach to the Mellin transforms of products and convolutions: In transform theory (be it Fourier, Laplace, Mellin or other) products of functions generally have transforms given by convolution integrals; and conversely, a function defined as the suitable convolution of two other functions has transform given by the product of the transforms of these two functions.

In the case of the Mellin transform, these relationships take the following form. On the one hand

$$\mathcal{M}[f(x) \cdot g(x)](s) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{M}[f](r) \cdot \mathcal{M}[g](s-r)\,\mathrm{d}r \tag{26}$$

and, on the other hand, if $h(x) := \int_0^\infty f\left(\frac{x}{y}\right) \cdot g(y)\,\frac{\mathrm{d}y}{y}$ then

$$\mathcal{M}[h](s) = \mathcal{M}[f](s) \cdot \mathcal{M}[g](s) \quad . \tag{27}$$

In existing theory, the requirements of classical integrability mean that these results entail complicated strip-wise conditions on c and on s in terms of the strip-wise domain restrictions which apply for the Mellin transforms of f and g. In some cases, these may mean that the desired Mellin transform does not even exist classically; and in equation 26 there is a further requirement which arises from the need for classical integrability, namely a square-integrability condition on $f(x) \cdot x^{c-\frac{1}{2}}$ based upon a version of Parseval's theorem to the effect that $\frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} \mid \mathcal{M}[f](c+it) \mid^2 \,\mathrm{d}t = \int_0^\infty \mid f(x) \mid^2 \cdot x^{2c-1}\,\mathrm{d}x$.

However, in the usual way, we may omit these technical considerations here since we are working within the generalised Césaro framework, with the only requirements thus being the need for generalised Césaro integrability and in particular the need to avoid pure log-divergences in the Césaro integrals defining the respective Mellin transforms.

The Mellin transform of a product of functions: If we now consider the case of a product of functions, $f(x) \cdot g(x)$, calculation of its Mellin transform by invoking equation 26 and performing the contour integration on the RHS, may still be challenging. As such, it is worth exploring how we might instead approach the calculation of $\mathcal{M}[f(x) \cdot g(x)](s)$ by alternative means using Taylor-series-to-the-left methods. Essentially the idea is as follows.

Suppose we know the TLA-coefficient functions of f and g. This might be based on knowing their individual Mellin transforms and applying our canonical relationship equation 6, or else simply from consideration of $f(x)$ and $g(x)$ themselves, using their power series near 0 and ∞ together with lemmas 1a-1c from [XI] to infer the correct canonical forms.

Suppose further that in each case we know how $\overset{\vee}{f}(m)$ splits into $f_0(m)$ and $f_\infty(m)$ for $m \in \mathbb{Z}$; and how $\overset{\vee}{g}(m)$ splits into $g_0(m)$ and $g_\infty(m)$.

Using the relationship proved in [XII] that for $h(x) := f(x) \cdot g(x)$ we have (check??)

$$\overset{\vee}{h}(m) = \sum_{j=-\infty}^{\infty} f_0(j) \cdot g_0(m-j) - \sum_{j=-\infty}^{\infty} f_\infty(j) \cdot g_\infty(m-j) \tag{28}$$

we can then try to replace m with general s to infer $\overset{\vee}{h}(s)$. If successful we can then finally derive $\mathcal{M}[h](s)$ from the canonical relationship that $\mathcal{M}[h](s) = \overset{\vee}{h}(-s) \cdot \frac{2\pi}{\sin(2\pi s)}$.

Let us see how this program plays out and how it opens up new approaches and results, by considering a progression of three elementary examples.

Example (v)(i) [$f(x) = \mathrm{e}^{-bx^2}$, $g(x) = \mathrm{e}^{-x^3}$]: In this case $h(x) = \mathrm{e}^{-x^3-bx^2}$ and calculating $\mathcal{M}[h](s)$ either using equation 26 or directly as $\int_0^\infty \mathrm{e}^{-x^3-bx^2} \cdot x^{s-1}\,\mathrm{d}x$ would appear challenging.

However, recall that $\mathcal{M}[\mathrm{e}^{-x}](s) = \Gamma(s)$. It follows from equation 16 and another elementary property of $\mathcal{M}$ that $\mathcal{M}[x^\nu f(x)](s) = \mathcal{M}[f(x)](s+\nu)$, that therefore $\mathcal{M}[\mathrm{e}^{-x^3} \cdot x^{2l}](s) = \frac{1}{3}\Gamma\left(\frac{s+2l}{3}\right)$ for all $l \in \mathbb{Z}_{\geq 0}$. Since we may expand e^{-bx^2} as $1 - bx^2 + \frac{b^2}{2!}x^4 - \ldots$, we can thus obtain $\mathcal{M}[h](s)$ as

$$\mathcal{M}[h](s) = \frac{1}{3}\sum_{l=0}^{\infty}(-1)^l \frac{b^l}{l!} \cdot \Gamma\left(\frac{s+2l}{3}\right) \tag{29}$$

which converges classically for any $s + 2l \notin 3 \cdot \mathbb{Z}_{\leq 0}$.

How can we obtain this using our alternative program? Well, in this case life is simplified by the fact that $f(x)$ and $g(x)$ are both Schwartzian as $x \to \infty$, so that we need only consider f_0 and g_0 in equation 28.

Now we already know $\overset{\vee}{f}(s)$ and $\overset{\vee}{g}(s)$, either from direct considerations in [XI] or from their Mellin transforms as calculated earlier in this section. We have:

$$\overset{\vee}{f}(s) = b^{\frac{s}{2}} \cdot \frac{1}{\left(\frac{s}{2}\right)!} \cdot \cos(\pi \frac{s}{2}) \cdot \cos(\pi s)$$

and

$$\overset{\vee}{g}(s) = \frac{1}{3} \cdot \frac{\sin(\pi s)}{\sin\left(\frac{\pi s}{3}\right)} \cdot \frac{1}{\left(\frac{s}{3}\right)!} \cdot \cos(\pi s) \quad .$$

Since $\overset{\vee}{f}(j) = 0$ for all $j \in \mathbb{Z}_{\leq 0}$ and also for all j a positive odd integer, it follows from equation 28 that on writing $j = 2l$ we have

$$\overset{\vee}{h}(m) = \sum_{l=0}^{\infty} (-1)^l \frac{b^l}{l!} \cdot \frac{1}{3} \cdot \frac{\sin(m\pi)}{\sin\left((m-2l)\frac{\pi}{3}\right)} \cdot \frac{1}{\left(\frac{(m-2l)}{3}\right)!} \cdot \cos(\pi m) \quad .$$

On replacing m with general s and recalling the functional equation for Γ (that $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$), this simplifies immediately to give

$$\overset{\vee}{h}(s) = \frac{1}{3} \sum_{l=0}^{\infty} (-1)^{l+1} \frac{b^l}{l!} \cdot \Gamma\left(\frac{2l-s}{3}\right) \cdot \frac{\sin(2\pi s)}{2\pi}$$

and we then recover equation 29 for $\mathcal{M}[h](s)$ immediately from the canonical relationship that

$$\mathcal{M}[h](s) = \overset{\vee}{h}(-s) \cdot \frac{2\pi}{\sin(2\pi s)} \quad . \tag{30}$$

Thus our alternative program does succeed here, albeit that in this case it only recovers a result that could be derived more directly by existing methods.

Example (v)(ii) [$h(x) = \frac{\mathrm{e}^{-x}}{1+x}$ and more generally $h(x) = \frac{\mathrm{e}^{-\mu x}}{x+\beta}$]: Consider first the special case of $h(x) = \frac{\mathrm{e}^{-x}}{1+x}$. Again, trying to calculate $\mathcal{M}[h](s)$ either from equation 26 or directly as $\int_0^\infty \frac{\mathrm{e}^{-x}}{1+x} \cdot x^{s-1}\,\mathrm{d}x$, is challenging.

Trying the same term-by-term classical approach: This time, however, performing a Taylor series expansion and working term by term is also problematic, no matter which way we attempt it.

If we expand $\frac{1}{1+x}$ as $1 - x + x^2 - x^3 + \ldots$ we get formally that

$$\mathcal{M}[h](s) = \sum_{j=0}^{\infty} (-1)^j \Gamma(s+j)$$

but this is highly non-convergent (either classically or in a generalised Césaro sense) for any s.

On the other hand, if we expand e^{-x} as $\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}x^j$ we get formally that

$$\mathcal{M}[h](s)=\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\cdot\frac{\pi}{\sin(\pi(s+j))}=\sum_{j=0}^{\infty}\frac{\pi}{\sin(\pi s)}\cdot\frac{1}{j!}=\frac{\mathrm{e}\pi}{\sin(\pi s)} \quad .$$

But this is also clearly wrong. It would imply that

$$\overset{\vee}{h}(s)=-\mathcal{M}[h](-s.\frac{\sin(2\pi s)}{2\pi}=\mathrm{e}\cos(\pi s)$$

and this cannot be correct, since $\frac{\mathrm{e}^{-x}}{1+x}$ is Schwartzian near ∞ and so should not have any negative powers of x with non-zero coefficients (i.e. we should have $\overset{\vee}{h}(j)=0$ for all $j\in\mathbb{Z}_{<0}$ and this is not the case under this formula for $\overset{\vee}{h}(s)$). Likewise the Taylor series for $\frac{\mathrm{e}^{-x}}{1+x}$ near 0 is $1-2x+\frac{5}{2}x^2-\ldots$, so that the coefficient of x^m for $m\in\mathbb{Z}_{\geq 0}$ is not $\mathrm{e}\cos(\pi m)$ but rather $\cos(\pi m)\cdot\sum_{0\leq j\leq m}\frac{1}{j!}$.

Why this approach fails: Thus the adaptation of traditional techniques which worked in the last example fails here. Why is this?

The answer is that, each in their own way, both these attempts fail to properly isolate and incorporate the power series behaviour of $\frac{1}{1+x}$ near ∞ as well as near 0. In the second approach we considered, if we expand e^{-x} as $\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}x^j$ we end up adding up the term by term Mellin transforms of $\frac{x^j}{1+x}$. Classically, however, each of these exists only for $-j<Re(s)<-j+1$ and extending outside this strip by generalised Césaro methods requires the application of a regular polynomial $q_j(s;P)$.

But, for any given $s\in\mathbb{C}$, $\frac{x^{j+s-1}}{1+x}$ has power series expansion near ∞ of $x^{j+s-2}-x^{j+s-3}+x^{j+s-4}-\ldots$. It follows that as j increases there are more and more classically divergent powers of x requiring Césaro-annihilation. Thus the degree of $q_j(s;P)$ grows without bound, and $q_j(s;P)$ acquires roots which accumulate densely on $\lambda=1$. Since we need to add up over all $j\in\mathbb{Z}_{\geq 0}$ *simultaneously*, it follows that we cannot perform the required reversal of summation and Mellin transform via generalised Césaro means using a single, common polynomial $q(s;P)$. This could perhaps be overcome by performing a proper Césaro array calculation, as outlined in [IV]-[VI], but this would be at a minimum both long and messy and we omit any consideration here.

On the other hand, as we saw, these problems do not arise in our first approach where we instead expand $\frac{1}{1+x}$ as $1-x+x^2-x^3+\ldots$, since $\mathrm{e}^{-x}\cdot x^j$ is still Schwartzian as $x\to\infty$ for any $j\in\mathbb{Z}_{\geq 0}$. But the resulting sum of Mellin transforms is then highly divergent for all s, and we will see below that this is likewise related to the fact that this Taylor series expansion of $g(x):=\frac{1}{1+x}$ only represents its power series expansion near 0[5], and thus neglects the g_∞ components in the power series make-up of g.

[5] with radius of convergence only 1, which cannot be extended by Césaro means

Correcting things using Taylor-series-to-the-left methods: Let us now instead try to obtain $\mathcal{M}[h](s)$ using our new program, taking $f(x) := \mathrm{e}^{-x}$ and $g(x) := \frac{1}{1+x}$ and being careful in our handling of power series components. We have

$$\check{f}(s) = \frac{\cos(\pi s)}{s!} \qquad \text{and} \qquad \check{g}(s) = \cos(\pi s)$$

and these are split as follows. For $j \in \mathbb{Z}$, $\check{f}$ is attributed entirely to f_0, so that $f_0(j) = \frac{\cos(\pi j)}{j!}$ and $f_\infty(j) \equiv 0$; and for g we have

$$g_0(j) = \begin{cases} \cos(\pi j)\ , & j \in \mathbb{Z}_{\geq 0} \\ 0\ , & j \in \mathbb{Z}_{<0} \end{cases} \qquad \text{and} \qquad g_\infty(j) = \begin{cases} 0\ , & j \in \mathbb{Z}_{\geq 0} \\ \cos(\pi(j+1))\ , & j \in \mathbb{Z}_{<0} \end{cases}$$

Since $f_0(j) = 0$ for all $j \in \mathbb{Z}_{<0}$ it follows in equation 28 that, for any $m \in \mathbb{Z}$, we have

$$\check{h}(m) = \sum_{0 \leq j \leq m} \frac{\cos(\pi j)}{j!} \cdot \cos(\pi(m-j)) = \left\{ \sum_{0 \leq j \leq m} \frac{1}{j!} \right\} \cdot \cos(\pi m) \quad .$$

Note that, although g_∞ does not appear explicitly in this formula, recognising the split between g_0 and g_∞ is still critical since it imposes the requirement that $(m-j) \geq 0$, i.e. that $j \leq m$, in the sum over the product of f_0 and g_0 terms.

When $m \in \mathbb{Z}_{\geq 0}$ this agrees with what we noted earlier. To extend from such m to negative integer m and indeed to general s, we recall our usual remainder-Césaro way of recasting $\sum_{0 \leq j \leq m}$ as a difference of two remainder-sums. We have

$$\check{h}(m) = \left\{ R_{+,0}\left[\frac{1}{z!}\right](0) - R_{+}\left[\frac{1}{z!}\right](m) \right\} \cdot \cos(\pi m)$$

and so, noting that $R_{+,0}\left[\frac{1}{z!}\right](0) = \mathrm{e}$, we have in general that

$$\check{h}(s) = \left\{ \mathrm{e} - R_{+}\left[\frac{1}{z!}\right](s) \right\} \cdot \cos(\pi s) \qquad \text{for all } s \in \mathbb{C} \quad . \tag{31}$$

It follows by our canonical relationship that the Mellin transform of h is given by

$$\mathcal{M}[h](s) = \check{h}(-s) \cdot \frac{2\pi}{\sin(2\pi s)} = \left\{ \mathrm{e} - R_{+}\left[\frac{1}{z!}\right](-s) \right\} \cdot \frac{\pi}{\sin(\pi s)} \quad . \tag{32}$$

Note that the remainder sum in equations 31 and 32 is classically rapidly convergent.

Our new program thus gives a clean, closed-form expression for $\mathcal{M}[h](s)$ where traditional methods for deriving such a formula had proved challenging.

Corollaries: Note in passing that $\overset{\vee}{h}(-1) = 0$ tautologically, and by considering $\overset{\vee}{h}'(-1) = \lim_{\epsilon\to 0} \frac{\overset{\vee}{h}(-1+\epsilon)}{\epsilon}$, we can also calculate $\int_0^\infty \frac{\mathrm{e}^{-x}}{1+x}\,\mathrm{d}x$ as

$$\int_0^\infty \frac{\mathrm{e}^{-x}}{1+x}\,\mathrm{d}x \;=\; \sum_{j=0}^{\infty} \frac{\Gamma'(j+1)}{(\Gamma(j+1))^2} \;=\; \sum_{j=0}^{\infty} \frac{\psi(j+1)}{\Gamma(j+1)}$$

where $\psi(s) = \frac{\Gamma'(s)}{\Gamma(s)}$ is the usual di-gamma function[6]. Since it is well-known that, for $n \in \mathbb{Z}_{>0}$, we have $\psi(n) = \left\{\sum_{j=1}^{n} \frac{1}{j}\right\} - \gamma$, where $\gamma \approx 0.577$ is the Euler-Mascheroni constant, this sum is both rapidly convergent and very easily calculated to high precision with only a small number of terms.

Since, moreover, using a trivial substitution, we can also calculate $\int_0^\infty \frac{\mathrm{e}^{-x}}{1+x}\,\mathrm{d}x$ in terms of the exponential integral function, $Ei(z)$, as $-\mathrm{e}\cdot Ei(-1)$, we get that

$$Ei(-1) = -\frac{1}{\mathrm{e}} \cdot \sum_{j=0}^{\infty} \frac{\psi(j+1)}{\Gamma(j+1)}$$

and this provides an interesting relationship between the functions $Ei(z)$, $\psi(z)$ and $\Gamma(z)$ at this special value.

In fact, a general such relationship is easily derived if we move from the special case of $h(x) = \frac{\mathrm{e}^{x}}{1+x}$ to the more general case of $h(x) = \frac{\mathrm{e}^{-\mu x}}{x+\beta}$.

The case of $h(x) = \frac{\mathrm{e}^{-\mu x}}{x+\beta}$: Applying our program in identical fashion to $f(x) = \mathrm{e}^{-\mu x}$ and $g(x) = \frac{1}{x+\beta}$, we have

$$\overset{\vee}{f}(s) = \cos(\pi s)\cdot\frac{\mu^s}{s!} \qquad \text{and} \qquad \overset{\vee}{g}(s) = \frac{\cos(\pi s)}{\beta^{s+1}}$$

with the same breakdown between g_0 and g_∞. We get that

$$\overset{\vee}{h}(s) = \left\{\mathrm{e}^{\mu\beta} - R_+\left[\frac{(\mu\beta)^z}{z!}\right](s)\right\}\cdot\frac{\cos(\pi s)}{\beta^{s+1}}$$

and

$$\mathcal{M}[h](s) = \left\{\mathrm{e}^{\mu\beta} - R_+\left[\frac{(\mu\beta)^z}{z!}\right](-s)\right\}\cdot\frac{\pi}{\sin(\pi s)}\cdot\frac{1}{\beta^{s+1}}$$

and it follows as before that

$$\int_0^\infty \frac{\mathrm{e}^{-\mu x}}{x+\beta}\,\mathrm{d}x \;=\; -\mathrm{e}^{\mu\beta}Ei(-\mu\beta) \;=\; \left\{-\ln(\mu\beta)\mathrm{e}^{\mu\beta} + R_{+,0}\left[\frac{\psi(z+1)(\mu\beta)^z}{\Gamma(z+1)}\right](0)\right\} \tag{33}$$

[6]Note here that there is no geometric component of the derivative of $R_+\left[\frac{1}{z!}\right](s)$ when performed in a generalised Césaro sense (cf discussion in [II]), so that we can take the derivative inside the remainder sum as $\frac{d}{ds}R_+\left[\frac{1}{z!}\right](s) = R_+\left[\frac{d}{dz}\frac{1}{\Gamma(z+1)}\right](s) = -R_+\left[\frac{\psi(z+1)}{\Gamma(z+1)}\right](s)$.

from which we derive in general the relationship that

$$Ei(-\nu) = -\mathrm{e}^{-\nu}\left\{-\ln(\nu)\mathrm{e}^{\nu} + R_{+,0}\left[\frac{\psi(z+1)\cdot\nu^z}{\Gamma(z+1)}\right](0)\right\} \quad . \tag{34}$$

The remainder sum in this relationship is classically convergent and equation 34 gives an interesting alternative series for $Ei(\nu)$ for arbitrary ν - one which complements the usual asymptotic series associated with the function. Equation 33 is easily validated numerically for random selections of μ and β, and shows that once again our new approach gives the correct, closed-form expressions for $\overset{\vee}{h}(s)$ and $\mathcal{M}[h](s)$ - on this occasion in a case where traditional methods had struggled.

Our last example gives another instance where this new program bears fruit, this time in a case where $\overset{\vee}{f}(m)$ and $\overset{\vee}{g}(m)$ both have non-trivial components arising from power series contributions near ∞ as well as near 0. It thereby illustrates in full how attention to delineating f_0 from f_∞, and g_0 from g_∞, underpins the success of this new approach.

Example (v)(iii) [$h(x) = \frac{1}{(1+x)(1+x^2)}$]: Here again, on taking $f(x) = \frac{1}{1+x^2}$ and $g(x) = \frac{1}{1+x}$, any effort to calculate $\mathcal{M}[h](s)$ by expanding either of f and g in a Taylor series around 0 and then working term-by-term runs into the same convergence problems as in example (v)(ii), and would at the very least require a careful and messy Césaro array analysis to overcome.

However, a direct calculation of $\mathcal{M}[h](s)$ is possible, based on a partial-fractions decomposition of $h(x)$ as $\frac{1}{2}\frac{1}{1+x} + \frac{-\frac{1}{2}x+\frac{1}{2}}{1+x^2}$ and the linearity of the operator $\mathcal{M}$. It yields immediately that

$$\mathcal{M}[h](s) = \frac{1}{2}\frac{\pi}{\sin(\pi s)} + \frac{1}{2}\frac{\frac{\pi}{2}}{\sin(\frac{\pi s}{2})} - \frac{1}{2}\frac{\frac{\pi}{2}}{\cos(\frac{\pi s}{2})} \tag{35}$$

and thus

$$\overset{\vee}{h}(s) = \frac{1}{2}\cos(\pi s) + \frac{1}{2}\cos(\pi s)\cdot\cos(\frac{\pi s}{2}) + \frac{1}{2}\cos(\pi s)\cdot\sin(\frac{\pi s}{2}) \quad . \tag{36}$$

Let us now show how we can reach the same results using our new program. In so doing, we will demonstrate how the presence of both f_0 and f_∞ (resp. g_0 and g_∞) components flows through this approach.

Taylor-series-to-the-left calculation: We have

$$\overset{\vee}{f}(s) = \cos(\pi s)\cdot\cos(\frac{\pi s}{2})$$

with

$$f_0(j) = \begin{cases} \cos(\pi j)\cdot\cos(\frac{\pi j}{2})\ , & j\in\mathbb{Z}_{\geq 0} \\ 0\ , & j\in\mathbb{Z}_{<0} \end{cases}$$

and

$$f_\infty(j) = \begin{cases} 0\ , & j \in \mathbb{Z}_{\geq 0} \\ -\cos(\pi j)\cdot\cos(\frac{\pi j}{2})\ , & j \in \mathbb{Z}_{<0} \end{cases}$$

and likewise

$$\overset{\vee}{g}(s) = \cos(\pi s)$$

with

$$g_0(j) = \begin{cases} \cos(\pi j)\ , & j \in \mathbb{Z}_{\geq 0} \\ 0\ , & j \in \mathbb{Z}_{<0} \end{cases} \qquad \text{and} \qquad g_\infty(j) = \begin{cases} 0\ , & j \in \mathbb{Z}_{\geq 0} \\ -\cos(\pi j)\ , & j \in \mathbb{Z}_{<0} \end{cases} \ .$$

Thus, for $m \in \mathbb{Z}_{\geq 0}$, we have as before from equation 28 that

$$\overset{\vee}{h}(m) \;=\; \sum_{0\leq j\leq m} f_0(j)\cdot g_0(m-j) \;=\; \sum_{0\leq j\leq m} \cos(\frac{\pi j}{2})\cdot\cos(\pi m)$$

on noting that there is no contribution from any product of f_∞ and g_∞ terms because there is no $j \in \mathbb{Z}_{<0}$ for which $(m-j)$ is also in $\mathbb{Z}_{<0}$.

In the same way, for $m \in \mathbb{Z}_{<0}$, we get no contribution from any product of f_0 and g_0 terms (since there is no $j \in \mathbb{Z}_{\geq 0}$ such that $(m-j)$ is also in $\mathbb{Z}_{\geq 0}$), and so

$$\overset{\vee}{h}(m) = -\sum_{m<j<0} \cos(\frac{\pi j}{2})\cdot\cos(\pi m) \quad .$$

Since the expression for $\overset{\vee}{h}(m)$ in both cases can be rewritten in the form

$$\overset{\vee}{h}(m) \;=\; \left\{R_{+,0}\left[\cos\left(\frac{\pi z}{2}\right)\right](0) - R_{+}\left[\cos\left(\frac{\pi z}{2}\right)\right](m)\right\}\cdot\cos(\pi m)$$

we can generalise immediately from $m \in \mathbb{Z}$ to arbitrary $s \in \mathbb{C}$ to obtain that

$$\overset{\vee}{h}(s) \;=\; \left\{R_{+,0}\left[\cos\left(\frac{\pi z}{2}\right)\right](0) - R_{+}\left[\cos\left(\frac{\pi z}{2}\right)\right](s)\right\}\cdot\cos(\pi s) \quad . \tag{37}$$

From this we deduce at once in the usual way that

$$\mathcal{M}[h](s) \;=\; \left\{R_{+,0}\left[\cos\left(\frac{\pi z}{2}\right)\right](0) - R_{+}\left[\cos\left(\frac{\pi z}{2}\right)\right](-s)\right\}\cdot\frac{\pi}{\sin(\pi s)} \tag{38}$$

and we see that our program again succeeds in yielding closed-form expressions for $\overset{\vee}{h}(s)$ and $\mathcal{M}[h](s)$ via consideration of TLA-coefficient functions and the use of Taylor-series-to-the-left methods for handling products.

To see the equivalence between these expressions and the earlier ones we derived directly, note that the remainder sums in equations 37 and 38 are all generalised Césaro sums. The summand function $\cos\left(\frac{\pi z}{2}\right)$ is oscillatory on a horizontal ray starting at any point $s \in \mathbb{C}$ and so these sums are all *strongly* Césaro convergent under a single application of P. We have that

$$R_{+,0}\left[\cos\left(\frac{\pi z}{2}\right)\right](0) = \frac{1}{2}$$

and on noting that $\cos\left(\frac{\pi(s+n)}{2}\right) = \cos\left(\frac{\pi s}{2}\right)\cdot\cos\left(\frac{\pi n}{2}\right) - \sin\left(\frac{\pi s}{2}\right)\cdot\sin\left(\frac{\pi n}{2}\right)$ we have in the same fashion that

$$R_+\left[\cos\left(\frac{\pi z}{2}\right)\right](s) = -\frac{1}{2}\cos\left(\frac{\pi s}{2}\right) - \frac{1}{2}\sin\left(\frac{\pi s}{2}\right) \quad .$$

It follows in equation 37 that we get

$$\overset{\vee}{h}(s) = \frac{1}{2}\cos(\pi s) + \frac{1}{2}\cos(\pi s)\cdot\cos(\frac{\pi s}{2}) + \frac{1}{2}\cos(\pi s)\cdot\sin(\frac{\pi s}{2})$$

which agrees with our earlier derivation in equation 36. Equations 38 and 35 then automatically agree because they both simply relate $\mathcal{M}[h](s)$ to $\overset{\vee}{h}(s)$ via our canonical relationship.

3.1 Final thoughts

The succession of three examples just given in the last section illustrates the effectiveness of Taylor-series-to-the-left concepts and methods for calculating Mellin transforms and TLA-coefficient functions where the function in question is a product of two or more other functions. It shows in particular how these methods can succeed where traditional techniques may struggle; the critical role played by careful attention to the split between contributions from power series expansions near 0 and near ∞ in the constituent functions in the product; and how these ideas and methods all naturally live within the world of generalised geometric Césaro convergence and remainder-summation, rather than merely the world of classical convergence.

Since, however, this paper has likely already "sat here too long"[7], we shall curtail things at this juncture and not undertake any more examples or illustrate these techniques further here.

We hope, after this paper and the previous explorations of [XI] and [XII], that the reader is by now convinced of the value of Taylor-series-to-the-left methods in opening new avenues for understanding and calculation regarding definite integrals, Mellin transforms, the asymptotic behaviour of functions, and many other allied mathematical issues - and the remaining two papers in this set will provide many further examples and applications.

We thus instead close with one final, large-scale observation. This is that, as noted from the outset in [I], generalised geometric Césaro convergence is just one among many possible generalised convergence schemes, each based on a suitable, regular operator analogous to P. In the same way, the Mellin transform is just one out of many possible transforms, such as Fourier transforms, Laplace transforms and so on.

In the same way that the Mellin transform naturally sits within the world of generalised Césaro convergence and has core properties reflecting the commutation properties of P, so it should be the case that these other transforms

[7] as Cromwell might have remarked had he lived 373 years later and been more interested in Mellin transforms and complex functions than history has thus far credited him with being

(which can often be rendered equivalent to each other under suitable changes of variables) should also each have a particular generalised convergence scheme within which it should live. Within this framework the domain of applicability of each should be able to be extended; many technical constraints on its application be relaxed and key theorems simplified; new connections to various series expressions be developed; and altogether an extension of ideas and methods analogous to what we have performed here and in [XI] and [XII] regarding the case of Mellin transforms and generalised Césaro theory be performed.

Explorations along these lines would, in our opinion, be both very interesting and eminently achievable. As Professor Atkins once observed (pers. comm.):

Down from the highlands and through the grey gloam,
The transforms cry out as they seek their true home!

We encourage readers to have at it along these lines!

4 Acknowledgements

We thank Professor Tommy Atkins and Professor Lord Protector Oliver Cromwell for many helpful insights (pers. comm.), and Professor T. Abby for his help in preparing this paper.

References

[I] R. Stone, *Introduction to generalised Césaro convergence I*, 2026

[II] R. Stone, *Introduction to generalised Césaro convergence II*, 2026

[III] R. Stone, *Introduction to generalised Césaro convergence III*, 2026

[IV] R. Stone, *Césaro Arrays I*, 2026

[V] R. Stone, *Césaro Arrays II*, 2026

[VI] R. Stone, *Césaro Arrays III*, 2026

[VII] R. Stone, *Root Identities I*, 2026

[VIII] R. Stone, *Root Identities II: Root identities for ζ - Part A*, 2026

[IX] R. Stone, *Root Identities II: Root identities for ζ - Part B*, 2026

[X] R. Stone, *Root Identities II: Root identities for ζ - Part C*, 2026

[XI] R. Stone, *Taylor series to the left I - Integration*, 2026

[XII] R. Stone, *Taylor series to the left II - behaviour approaching infinity*, 2026

[1] H.M. Edwards, *Riemann's Zeta Function*, Academic Press, 1974