

Taylor Series to the Left IV - Seeing to the edge of the universe with a microscope

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Abstract

We derive Stirling's theorem on the asymptotic behaviour of the Gamma function, $\Gamma(z + 1)$, as $z \rightarrow \infty$ (i.e. as we move towards the edge of the universe) purely from a detailed consideration of its Taylor series near 0 (i.e. by looking under higher and higher magnification through a microscope focussed on an infinitesimal neighbourhood of $z = 0$). This gives a non-trivial example of local-to-global inference based on Taylor-series-to-the-left methods. Central to the approach are TLA-coefficient functions and their relationship on the one hand to the simultaneous power series make-up of a function near 0 and near ∞ and, on the other, to its Mellin transform defined under a generalised Césaro framework. The reasoning is applicable much more broadly than just for Γ , and is extremely straightforward and direct. It almost amounts simply to reading off the terms in the asymptotic expansion for $f(z)$ as $z \rightarrow \infty$ straight from the formula for the derivatives $f^{(n)}(0)$, $n \in \mathbb{Z}_{\geq 0}$.

1 Introduction

In [XI]-[XIII] we introduced Taylor-series-to-the-left methods based on the TLA-coefficient function, $\check{f}(s)$, of a function $f(x)$.

In particular, in [XII] we introduced a conjecture relating $\check{f}(s)$ *simultaneously* to the power series expansions for $f(x)$ near 0 and near ∞ , and we illustrated it with various elementary and non-elementary examples. We further discussed how this simultaneous capture of information about the power-series for $f(x)$ near 0 and ∞ opens up the possibility of local-to-global inference - deducing the power series expansion for $f(x)$ as $x \rightarrow \infty$ from a knowledge of the general form of $\check{f}(s)$ and a full understanding of the power series for $f(x)$ near 0.

In [XIII] we then proved this conjecture using Mellin transforms and the canonical relationship between $\mathcal{M}[f](s)$ and $\check{f}(s)$, namely that

$$\check{f}(s) = -\mathcal{M}[f](-s) \cdot \frac{\sin(2\pi s)}{2\pi} \quad \text{and thus} \quad \mathcal{M}[f](s) = \check{f}(-s) \cdot \frac{2\pi}{\sin(2\pi s)} \quad . \quad (1)$$

Here the definition of the Mellin transform has been extended to the setting of generalised Césaro, rather than merely classical, convergence - resulting in considerable simplification, as well as material expansion of the domain of applicability of the operator \mathcal{M} . This natural setting of Mellin transform theory within the generalised Césaro convergence framework was explored at length in [XIII], where the key relationship between poles of $\mathcal{M}[f](s)$ and terms of the form $x^\rho (\ln x)^{N-1}$, $N \in \mathbb{Z}_{>0}$, was affirmed.

In this short paper we bring all these aspects together. Our purpose is to show that, by doing so, we can derive Stirling's theorem giving the asymptotic behaviour of $\Gamma(z+1)$ as $z \rightarrow \infty$, using methods which seem trivial and which involve little more than reading off terms from a simple formula.

Although Stirling's theorem is, of course, well-known, it is non-trivial and much harder to prove by other means, so that this provides a good illustration of the power of Taylor-series-to-the-left methods in general, and their application in local-to-global inference in particular.

1.1 Final introductory notes

We thank Prof Morowbie Jukes of the Kafirstan Institute of Mathematics (KIM) for many helpful comments, and acknowledge the strange and riveting ride that he experienced in the course of discussing with us these ideas.

2 Derivation of Stirling's theorem using Taylor-series-to-the-left methods

As usual it is easier to work not with $\Gamma(z+1)$ directly, but rather with its logarithm, $\ln(\Gamma(z+1))$. We use two key results from [XI]-[XIII]. First, there is the following result, initially conjectured in [XII] and then proven in [XIII]:

Result 1: *Suppose f is a smooth function on $(0, \infty)$ and suppose that for x near 0, f has the power series expansion $\sum_{j=m_0}^{\infty} f_0(j)x^j$ for some integer m_0 ; and that as $x \rightarrow \infty$, f has the power series expansion $\sum_{j=-\infty}^{m_\infty} f_\infty(j)x^j$ for some integer m_∞ . Here, the power series of which $\{f_0(j)\}_{j=m_0}^{\infty}$ and $\{f_\infty(j)\}_{j=-\infty}^{m_\infty}$ are the coefficient sequences may be Taylor or Laurent series with some finite or infinite radius of convergence, R ; or may merely be asymptotic expansions not convergent for any x (so $R = 0$). Then if f is integrable in a generalised Césaro sense on $[0, \infty)$ it has a canonical Taylor-coefficient function, $\check{f}(s)$ ($s \in \mathbb{C}$), and this satisfies that*

$$\check{f}(m) = f_0(m) - f_\infty(m) \quad \text{for all } s = m \in \mathbb{Z}. \quad (2)$$

Secondly, we use the following result from [XIII] relating power and power-log divergences in the power series expansions for $f(x)$ near 0 and ∞ to the poles

and residues of $\mathcal{M}[f](s)$:

Result 2: *Suppose that f is a smooth function on $(0, \infty)$, integrable in a generalised Césaro sense on $[0, \infty)$, and suppose f has power series expansions as per result 1 for x near 0 and as $x \rightarrow \infty$. Then for any $m \in \mathbb{Z}$, if the power series expansion for f near 0 contains a term ax^m and the power series expansion for f as $x \rightarrow \infty$ contains a term bx^m , it follows that $\mathcal{M}[f](s)$ has a simple pole at $s = m$ with residue $(a - b)$, so that the Laurent-series expansion for $\mathcal{M}[f](s)$ in a neighbourhood of $s = m$ has a term $\frac{(a-b)}{s-m}$.*

In the same way, if for $N \in \mathbb{Z}_{>0}$ we allow f to contain terms of the form $\tilde{a}x^m(\ln x)^{N-1}$ and $\tilde{b}x^m(\ln x)^{N-1}$ in generalised power series expansions near 0 and ∞ respectively, then $\mathcal{M}[f](s)$ has a pole of order N at $s = m$, with coefficient $(-1)^{N-1} \cdot (N-1)! \cdot (\tilde{a} - \tilde{b})$, so that the Laurent-series expansion for $\mathcal{M}[f](s)$ in a neighbourhood of $s = m$ has a term $(-1)^{N-1} \cdot (N-1)! \cdot \frac{(\tilde{a}-\tilde{b})}{(s-m)^N}$.

Comments: (i) As discussed in [XIII], this result holds within the generalised Césaro convergence framework. It thus holds for functions, f , which may have singularities (i.e. negative power or power-log divergences) as $x \rightarrow 0$, and which may have non-negative power or power-log divergences as $x \rightarrow \infty$. This will be critical for the case of $\ln(\Gamma(z+1))$ we are interested in.

(ii) Based on the canonical relationship between $\mathcal{M}[f](s)$ and $\check{f}(s)$ given in equation 1, result 2 can be easily recast in terms of $\check{f}(s)$ as follows:

Result 2a: *Suppose that f is as given in result 2. If the power series expansion for f near 0 contains a term ax^m and the power series expansion for f as $x \rightarrow \infty$ contains a term bx^m , then the Laurent-series expansion for $\check{f}(s)$ in a neighbourhood of $s = m$ contains the constant term $(a - b)$.*

More generally, if for $N \in \mathbb{Z}_{>0}$ the generalised power series expansion for f near 0 contains a term $\tilde{a}x^m(\ln x)^{N-1}$ and the generalised power series expansion for f near ∞ contains a term $\tilde{b}x^m(\ln x)^{N-1}$, then the Laurent-series expansion for $\check{f}(s)$ in a neighbourhood of $s = m$ has a term $(N-1)! \cdot \frac{(\tilde{a}-\tilde{b})}{(s+m)^{N-1}}$.

2.1 Derivation of Stirling's theorem

Turning now to $f(z) := \ln(\Gamma(z+1))$, let us get out our microscopes and consider its behaviour in an infinitesimal neighbourhood of $z = 0$. We have that

Result 3a: $f(z) := \ln(\Gamma(z+1))$ has the Taylor series

$$\ln(\Gamma(z+1)) = -\gamma z + \sum_{j=2}^{\infty} (-1)^j \frac{\zeta(j)}{j} z^j \quad (3)$$

around 0. This has radius of convergence $R = 1$ and thus converges classically for all $|z| < 1$.

This can either be taken as our starting-point, since it is a well-known result; or alternatively we can recall that we proved this in [II] based on our characterisation of $\ln(\Gamma(z+1))$ using generalised geometric Césaro remainder sums as $\ln(\Gamma(z+1)) = R_+[\ln \tilde{z}](0) - R_+[\ln \tilde{z}](z)^1$.

Equation 3 says that for $m \in \mathbb{Z}_{\geq 2}$ we have

$$f_0(m) = \frac{\zeta(m)}{m} \cdot \cos(\pi m) \quad (4)$$

and since this holds for m arbitrarily large this should also represent $\check{f}(m)$ for all m sufficiently large. We therefore infer that in general $\check{f}(s)$ is given by

$$\check{f}(s) = \frac{\zeta(s)}{s} \cdot \cos(\pi s) \quad \text{for all } s \in \mathbb{C} \quad (5)$$

Now $\check{f}(s)$ is then singular at both $s = 1$ and $s = 0$, whereas $f_0(1) = -\gamma$ and $f_0(0) = 0$. It is tempting to speculate that perhaps the $-\gamma$ in $f_0(1)$ still represents $\check{f}(1)$ but with some sort of "renormalized" evaluation of $\zeta(1)$, but this would be an error². The singularities in $\check{f}(s)$ at $s = 1$ and $s = 0$ actually reflect critical aspects of the divergent asymptotic behaviour of $\ln(\Gamma(z+1))$ and so need to be examined carefully.

To this end, for $s = 1 + \epsilon$ near 1, we have that $\zeta(1 + \epsilon) = \frac{1}{\epsilon} + \gamma + O(\epsilon)$ and $\cos(\pi(1 + \epsilon)) = -1 + O(\epsilon^2)$ and $\frac{1}{1+\epsilon} = 1 - \epsilon + O(\epsilon^2)$, so that

$$\check{f}(s) = \frac{\zeta(s)}{s} \cdot \cos(\pi s) \approx \frac{-1}{s-1} + (-\gamma + 1) + O(\epsilon) \quad (6)$$

And in the same way, for $s = \epsilon$ near 0, we have that $\zeta(\epsilon) = \zeta(0) + \zeta'(0)\epsilon + O(\epsilon^2) = -\frac{1}{2} - \frac{1}{2} \ln(2\pi)\epsilon + O(\epsilon^2)$ and $\cos(\pi\epsilon) = 1 + O(\epsilon^2)$, so that

$$\check{f}(s) = \frac{\zeta(s)}{s} \cdot \cos(\pi s) \approx \frac{-1}{2} \frac{1}{s} - \frac{1}{2} \ln(2\pi) + O(\epsilon) \quad (7)$$

As for $s = -n \in \mathbb{Z}_{<0}$, we have that $\check{f}(-n)$ is non-singular and is given by

$$\check{f}(-n) = (-1)^{n+1} \cdot \frac{\zeta(-n)}{n} \quad (8)$$

¹Using this characterisation and the fact that $\ln(n + \epsilon) = \ln n + \frac{\epsilon}{n} - \frac{1}{2} \frac{\epsilon^2}{n^2} + \frac{1}{3} \frac{\epsilon^3}{n^3} - \dots$, the components of the Taylor series in equation 3 for powers z^j , $j \geq 2$, follow at once. For $j = 1$ we need to be careful to remember the *geometric* aspects of our remainder sums in the calculation of $\frac{d}{dz} \ln(\Gamma(z+1))|_{z=0} = \lim_{h \rightarrow 0} \frac{\{R_+[\ln \tilde{z}](0) - R_+[\ln \tilde{z}](h)\}}{h}$ and thus place our summands at the right geometric locations. Doing this leads to the cancellation of the naive log-divergence which otherwise appears, leaving only the residual $-\gamma$.

²As Prof Jukes remarked to us (pers. comm.) "perhaps not as grave an error as going riding near the Sutlej after midnight with a high fever and falling into a well-guarded leper colony, but an error nonetheless".

But now recall result 2a. The simple pole and constant term in equation 6 near $s = 1$ mean there are terms $\tilde{a}z \ln z$ and $\tilde{b}z \ln z$ in the power series for $\ln(\Gamma(z+1))$ near 0 and ∞ (resp.) such that $(\tilde{a} - \tilde{b}) = -1$; and likewise that there are terms az and bz in these power series such that $a - b = -\gamma + 1$.

Since we know from equation 3 that the power series for $\ln(\Gamma(z+1))$ near 0 has no term of the form $\tilde{a}z \ln z$ and has a single term of the form $-\gamma z$, it follows that $\tilde{a} = 0$ and $a = -\gamma$, and thus $\tilde{b} = 1$ and $b = -1$. Thus the power series for $\ln(\Gamma(z+1))$ as $z \rightarrow \infty$ contains the divergent terms $z \ln z - z$.

Similarly, from considering the simple pole and constant term in equation 7 for s near 0, and noting that the power series for $\ln(\Gamma(z+1))$ near 0 contains no term of the form $\tilde{a} \ln z$ and no constant term, it follows that the power series for $\ln(\Gamma(z+1))$ as $z \rightarrow \infty$ contains the terms $\frac{1}{2} \ln z + \frac{1}{2} \ln(2\pi)$.

And lastly, from equation 8 and the fact that there are no negative powers in the power series for $\ln(\Gamma(z+1))$ near 0, it follows that the power series for $\ln(\Gamma(z+1))$ as $z \rightarrow \infty$ also contains the decaying asymptotic series

$$\sum_{n=1}^{\infty} (-1)^n \cdot \frac{\zeta(-n)}{n} z^{-n} = \frac{1}{12} \frac{1}{z} - \frac{1}{360} \frac{1}{z^3} + \frac{1}{1260} \frac{1}{z^5} - \dots \quad .$$

Combining these deductions, we obtain finally that the power series for $\ln(\Gamma(z+1))$ as $z \rightarrow \infty$ is given by

$$\ln(\Gamma(z+1)) \approx \left(z + \frac{1}{2}\right) \ln z - z + \frac{1}{2} \ln(2\pi) + \frac{1}{12} \frac{1}{z} - \frac{1}{360} \frac{1}{z^3} + \frac{1}{1260} \frac{1}{z^5} - \dots \quad (9)$$

and this is precisely Stirling's famous theorem giving the asymptotic behaviour of the Gamma function as $z \rightarrow \infty$.

It can of course be expressed directly in terms of Γ rather than the log-Gamma function by exponentiating this equation to obtain that

$$\Gamma(z+1) \approx \sqrt{2\pi} \cdot z^{z+\frac{1}{2}} \cdot e^{-z} \cdot \left\{ 1 + \frac{1}{12} \frac{1}{z} + \frac{1}{288} \frac{1}{z^2} - \frac{139}{51840} \frac{1}{z^3} + \dots \right\} \quad . \quad (10)$$

Final Comments: (i) Equation 9 is an asymptotic power series, not classically convergent for z in any neighbourhood of ∞ (i.e. for $u = \frac{1}{z}$ in any neighbourhood of 0). This is because, despite decaying initially, the coefficients $(-1)^n \cdot \frac{\zeta(-n)}{n}$ grow unboundedly as $n \rightarrow \infty$.

This illustrates how the Taylor-series-to-the-left methods and results developed in [XI]-[XIII] apply not only where the power series for $f(x)$ near 0 and near ∞ both have positive radius of convergence, but also where one or both is only an asymptotic series with $R = 0$.

Likewise, the presence of the $(z + \frac{1}{2}) \ln z$ and z divergences in equation 9 as $z \rightarrow \infty$ is a reminder that these methods work seamlessly in the generalised Césaro domain. By contrast, the Mellin transform calculations and other techniques we have employed here would break down - and our derivation of Stirling's theorem would fail - if we restricted ourselves to requiring classical

convergence.

(ii) This derivation of Stirling's theorem is as pure an example of non-trivial local-to-global inference as could be hoped for.

We have focussed our microscope on an infinitesimal neighbourhood of $z = 0$ to deduce the full sequence of derivatives, and hence the Taylor series, for $\ln(\Gamma(z + 1))$ near 0. Then, merely by reading off the coefficient values, $\check{f}(m)$, to the left as $m \rightarrow -\infty$, and by simple development of the singularities in $\check{f}(s)$ at $m = 1$ and $m = 0$, we have been able to look far out and fully reconstruct the asymptotic power series describing the behaviour of $\ln(\Gamma(z + 1))$ as z heads toward the edge of the universe (i.e. $z \rightarrow \infty$) - right down to such fine detail as why the constant term in equation 9 is $-\zeta'(0)$.

There is, of course, nothing in principle preventing us from also reasoning in the opposite direction. That is, we could combine the global character of $\check{f}(s)$ with a detailed knowledge of the power series for a function $f(x)$ near ∞ to deduce the power series describing the behaviour of $f(x)$ near $x = 0$.

In the ordinary course of events such "global-to-local inference" would naturally be less common, but there do exist interesting functions whose behaviour is more intelligible for large z than for small z , and whose complex power series behaviour near 0 might be rendered more understandable by such efforts to see down to the sub-atomic level with a telescope!

3 Acknowledgements

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