Taylor Series to the Left V - a Miscellany of Further Applications

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Abstract

We conclude our series on Taylor-series-to-the-left with a miscellany of further applications of TLA-coefficient functions and associated methods. First we consider the Mellin transforms of p-sum functions. We use a simple example to show the depth of their connection to the generalised Césaro convergence framework and hence discuss the broad applicability of our methods in this area of analytic number theory. Next we consider a number of new applications to integrals - including some altogether new definite integral evaluations (for example for $\int_0^\infty \mathrm{e}^{-p(x)}\,\mathrm{d}x$ for any polynomial $p(x)$ of arbitrary degree with positive leading coefficient), as well as some new connections to existing methods and results. These in turn lead in two further directions - first, to an extension to integrands with poles on the integration domain $[0,\infty)$; and secondly to a broader discussion of the possibility of a "calculus" for TLA-coefficient functions. Next, we discuss open areas where the TLA-coefficient methods we have developed still encounter obstacles. We give simple examples illustrating these issues, discuss some partial-explanations of the associated problems, and try to articulate where the remaining conundra lie.[1] Finally, we conclude with a mixture of speculations of varying wildness regarding future areas for exploration.

1 Introduction

In [XI]-[XIV] we introduced the concept of the TLA-coefficient function, $\overset{\vee}{f}(s)$, of a given function $f(x)$, and we developed a number of associated Taylor-series-to-the-left methods and results.

These include lemmas for deriving the correct canonical form of the TLA-coefficient function from among a class of "gauge-equivalent" choices; the relationship between this canonical form and the Mellin transform of f; integration results relating $\int_0^\infty f(x)\,\mathrm{d}x$ to $\overset{\vee}{f}'(-1)$ and associated methods for bootstrapping Mellin transforms more generally (or indeed calculating any definite integral

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[1] The plural of conundrum is conundra; were these to be posed in Siberia they would be conundra in the tundra!

with an integrand parametrised by a complex parameter) from this result; results relating $\overset{\vee}{f}(s)$ at the integer points $s = m \in \mathbb{Z}$ to the power series for f near 0 and ∞, thereby potentially facilitating local-to-global inference; some initial results regarding how TLA-coefficient functions behave when taking standard combinations (e.g. products) of the underlying functions; and miscellaneous other examples and results extending these.

Across all of these developments, an overarching theme has been the demonstration that the right context for all these results is not classical, but rather generalised Césaro, convergence - so that all Mellin transform theory, all definite integrals and all the results in [XI]-[XIV] should be understood as existing within the generalised Césaro convergence framework.

Here we conclude this set of papers on Taylor-series-to-the-left with a miscellany of further applications.

In section 2 we start by considering an important class of Mellin transforms - those of p-sum functions $\sigma(x)$. These arise critically in many areas of number theory, in particular the theory of the Riemann zeta function, $\zeta(s)$. Since p-sum functions on $[0, \infty)$ are identically zero in a neighbourhood of 0, the values of $\overset{\vee}{\sigma}(s)$ (or equivalently the residues of the poles of $\mathcal{M}[\sigma](s)$) at $s = m \in \mathbb{Z}$ will automatically tell us about the asymptotic behaviour of $\sigma(x)$ as $x \to \infty$, which is of acute interest.

In section 2.1 we first review a number of important such p-sum functions - $J(X)$, $\pi(X)$, $R(X)$ and $\psi(X)$ - and discuss how results concerning them and their Mellin transforms are currently understood, and how this might change in moving to the generalised Césaro convergence framework and applying Taylor-series-to-the-left methods.

After developing the necessary formulae for a p-sum function in general in section 2.2, we then consider in depth in section 2.3 a simple example, but one which clearly illustrates the connection between the TLA-coefficient function and associated Mellin transform of a p-sum function and its asymptotic behaviour - and demonstrates how this expresses itself in a completely new way within the generalised Césaro framework.

Based on this we then return in section 2.4 to the examples from section 2.1. We show that von Mangoldt's famous formula for $\psi(X)$ follows trivially under our Taylor-series-to-the-left approach, but now as a strong Césaro asymptotic result; and we discuss how this might in turn be coupled with our theory of Césaro arrays (from [IV]-[VI]) to find an alternative expression for $\psi(X)$. We briefly consider $J(X)$ and $\pi(X)$ in the same light and then also analyse a new example - that of the p-sum function of the Möbius function, which leads to an interesting new perspective. Finally, we consider briefly the Mellin transform of the p-sum function of a strongly-periodic sequence, and possible connections to the results of [VI].

In section 3 we then change tack and consider a series of new examples illustrating the application of our results in the calculation of definite integrals, $\int_0^\infty f(x)\,\mathrm{d}x$.

Having seen in [XI] the close connection of our key integration result to

Ramanujan's master theorem (RMT), our first example shows that it likewise provides an alternative approach for evaluating integrals which otherwise require Feynman's trick and the solution of a differential equation.

Our second example set then considers closed-form evaluation of integrals of the form $\int_0^\infty \frac{\mathrm{e}^{-x^n}}{1+x^2}\,\mathrm{d}x$, which would otherwise be difficult to calculate other than numerically.

Finally, in our third example set we derive a closed-form formula for $\int_0^\infty \mathrm{e}^{-p(x)}\,\mathrm{d}x$, where $p(x)$ is any polynomial of arbitrary degree with positive leading coefficient. Such formulae are well-known for the simplest cases of degree 1 and 2 and can be readily found in compendiums like [1], but the general formula we derive for degree $n \geq 3$ does not appear to be included in such places and may, indeed, be entirely new.

In section 4 we then extend the Taylor-series-to-the-left methodology we have developed for Césaro-convergent definite integrals $\int_0^\infty f(x)\,\mathrm{d}x$ in a different direction - namely to cover also the case where $f(x)$ has pole-singularities on the integration domain $[0,\infty)$. We show how generalised Césaro convergence, together with definition via the Cauchy principal value, leaves all our integration results from [XI]-[XIV] in tact. We show further that if we avoid issues at such poles by instead amending the integration contour to go around them, this simply corresponds within our approach to making a different gauge-choice for the TLA-coefficient function we use.

In some of the examples from section 3 (as in some examples in previous papers) we rely on what is effectively a "product rule" for obtaining the TLA-coefficient function of a product function from the TLA-coefficient functions of the individual functions in the product (i.e. for obtaining $\overset{\vee}{h}(s)$ from $\overset{\vee}{f}(s)$ and $\overset{\vee}{g}(s)$ where $h(x) := f(x) \cdot g(x)$). In section 5 we consider more generally whether this might merely constitute the first component of a broader set of such rules comprising a "calculus" of TLA-coefficient functions.

Such a calculus would be very valuable and might, in many cases, provide a more tractable set of rules to use for calculation as an alternative to the corresponding set of rules already known for Mellin transforms.

We begin some explorations in this direction and consider a number of examples. These raise the possibility that formal symbols and formal function elements (as introduced in [III]) may have a useful role to play in any such calculus.

In the course of these explorations, however, we soon come up against examples which demonstrate the limits of the theory we have developed so far in [XI]-[XIV]. In section 6 we consider one such example in detail and, based on it, we consider also a related example which highlights some of these difficulties in sharper relief.

Arising from this we note certain ideas which may partially resolve these difficulties, but we also articulate in detail where we feel these explanations remain unsatisfactory and where mysteries remain. We relate these issues to some broader speculations regarding the interpretation of the TLA-function in general and conclude with some thoughts on how we might proceed from here.

Finally, since this is the last in this set of papers on Taylor-series-to-the-left methods, we conclude in section 7 with a brief (and at times wildly speculative!) collection of stray thoughts and observations.[2] These cover where things stand and what possible future avenues of exploration might exist using these ideas and methods.

1.1 Final notes

We thank Prof Otis Yeere of teh Kafiristan Institute of Mathematics (KIM) and his mentor, Prof G.O.T.C. Headings for a number of helpful suggestions and reminders (pers. comm.); and we dedicate this paper to them for their many insights of enduring relevance.

2 Generalised Césaro convergence, and Mellin transforms of p-sum functions

Partial sum functions - on the one hand, their Mellin transforms and on the other, their asymptotic growth and behaviour as $x \to \infty$ - are of critical interest in analytic number theory and especially in the theory of the Riemann zeta function.

2.1 p-sum functions in the theory of ζ - examples and connections to Césaro methods

We begin by quickly reviewing three examples - the functions $J(x)$, $\pi(x)$, and $\psi(x)$ - summarising their importance and current treatment. In each case, we then make brief comment on how that treatment might be amended in light of our Taylor-series-to-the-left theory and the generalised Césaro framework. Readers familiar with these functions and their classical treatment can easily skip the summaries and just note the commentary attached to each case.

Example (1) [$J(x)$]: A central function in the theory of ζ is the function $J(x)$. Let $dJ(x)$ be the Stieltjes measure with weight $\frac{1}{n}$ at each prime power p^n, so that $J(x)$ is the p-sum function

$$J(x) := \sum_{p^n < x} \frac{1}{n} \quad .$$

Then by the Euler product formula, which says that

$$\log(\zeta(s)) = \sum_p \left\{ \sum_{n=1}^{\infty} \left(\frac{1}{n}\right) p^{-ns} \right\}$$

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[2] Actually, while some of the ideas canvassed are highly speculative, others are relatively well-founded and are buttressed by new example calculations including one, for example, investigating an interesting formula of Ramanujan.

we have that

$$\log(\zeta(s)) = \int_0^\infty x^{-s}\mathrm{d}J(x) = s \cdot \int_0^\infty J(x)\cdot x^{-s-1}\mathrm{d}x \quad (Re(s) > 1). \tag{1}$$

Thus $\frac{\log(\zeta(s))}{s}$ is the Mellin transform $\mathcal{M}[J](-s)^3$. Traditional Mellin inversion as

$$J(x) = \frac{1}{2\pi i}\cdot\int_{a-i\infty}^{a+i\infty} \frac{\log(\zeta(s))}{s} x^s \mathrm{d}s \quad (a > 1) \tag{2}$$

then gives us a formula for $J(x)$ in terms of $\log(\zeta(s))$.

But the functional equation for ζ together with the Hadamard expression for $\zeta(s)$ as a product over its roots (namely that $\xi(s) = \xi(0)\cdot\prod_{\rho\in NT}\left(1-\frac{s}{\rho}\right)$ where $\xi(s) := \left(\frac{s}{2}\right)!\cdot\pi^{-\frac{s}{2}}\cdot(s-1)\cdot\zeta(s)$ as usual) gives an alternative expression for $\log(\zeta(s))$ as

$$\log(\zeta(s)) = \log(\xi(0)) + \sum_{\rho\in NT}\log(1-\frac{s}{\rho}) - \log\left((\frac{s}{2})!\right) + \frac{s}{2}\log\pi - \log(s-1)\ . \tag{3}$$

Integrating by parts in equation 2 gives that

$$J(x) = -\frac{1}{2\pi i}\cdot\frac{1}{\log x}\cdot\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left[\frac{\log(\zeta(s))}{s}\right]x^s\mathrm{d}s \quad (a > 1) \tag{4}$$

and by than applying equation 3 Riemann deduced his famous formula that

$$\begin{aligned} J(x) &= Li(x) - \sum_{Im(\rho)>0}\left[Li(x^\rho) + Li(x^{1-\rho})\right] - \sum_{m=1}^{\infty} Li(x^{-2m}) - \log 2 \\ &= Li(x) - \sum_{Im(\rho)>0}\left[Li(x^\rho) + Li(x^{1-\rho})\right] + \int_x^\infty \frac{\mathrm{d}t}{t(t^2-1)\log t} - \log 2 \end{aligned} \tag{5}$$

for $x > 1$. Here $Li(x)$ is the Cauchy p.v. across $t = 1$ of the integral $\int_0^x \frac{1}{\log t}\,\mathrm{d}t$ and $Li(x^\rho)$ is interpreted as $Ei(\rho\log x)$.

Comment [Current understanding and extension]: We see that the p-sum function $J(x)$ and its Mellin transform are intimately connected with ζ and the process of Mellin inversion leads directly to deep results like equation 5 regarding the distribution of prime numbers.

Of course the derivation outlined above goes back to Riemann and uses traditional definitions and techniques for applying the Mellin operator and its inverse. This means it comes with all the attendant need for care re classical convergence and associated restrictions on domains for a, x and so forth which that entails.

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[3] or $\langle J(x) \mid x^s\rangle$ in the quasi-inner-product terminology we introduced in [XIII]

However, in light of our results in [XI] - [XIV] it would be interesting to consider how this might be alternately understood - and perhaps extended - in light of our extension of the natural setting of Mellin transform calculations from classical to generalised geometric Césaro convergence; and given the alternative pathway that now exists for performing Mellin inversion via TLA-coefficient functions rather than via line integrals like 2 or 4. We will discuss these possibilities in more detail after considering example 2.

Example (2) [$\pi(x)$]: Likewise, if we consider the p-sum function $\pi(x) := \sum_{p<x} 1$, which counts the primes less than x and is thus even more directly related to the distribution of primes, then we have

$$J(x) = \pi(x) + \frac{1}{2}\pi(x^{\frac{1}{2}}) + \frac{1}{3}\pi(x^{\frac{1}{3}}) + \ldots + \frac{1}{n}\pi(x^{\frac{1}{n}}) + \ldots \tag{6}$$

which is a finite sum for any given x. Möbius inversion thus leads to the expression

$$\begin{aligned} \pi(x) &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n} J(x^{\frac{1}{n}}) \\ &= J(x) - \frac{1}{2}J(x^{\frac{1}{2}}) - \frac{1}{3}J(x^{\frac{1}{3}}) - \frac{1}{5}J(x^{\frac{1}{5}}) + \frac{1}{6}J(x^{\frac{1}{6}}) - \frac{1}{7}J(x^{\frac{1}{7}}) - \ldots \end{aligned} \tag{7}$$

and by invoking result 5 and formally reversing the order of summation, we obtain Riemann's famous formula for $\pi(x)$, namely that for $x > 1$

$$\begin{aligned} \pi(x) &= R(x) - \sum_{Im(\rho)>0} \left[R(x^{\rho}) + R(x^{1-\rho})\right] - \sum_{m=1}^{\infty} R(x^{-2m}) \\ &= R(x) - \sum_{Im(\rho)>0} \left[R(x^{\rho}) + R(x^{1-\rho})\right] + \frac{1}{\pi}\tan^{-1}\left(\frac{\pi}{\log x}\right) - \frac{1}{\log x} \end{aligned} \tag{8}$$

Here $R(x)$ is formally defined by

$$R(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} Li(x^{\frac{1}{n}}) \tag{9}$$

which has the Gram's series expansion

$$R(x) = 1 + \sum_{k=1}^{\infty} \frac{(\log x)^k}{k \cdot k! \cdot \zeta(k+1)} \quad \text{for} \quad x > 1\,. \tag{10}$$

Comment [Current understanding and extension]: Here again, while the results sketched above are classical and long-established, it would be interesting to re-assess them in light of the generalised geometric Césaro framework developed in [XI] - [XIV]. For example, in equation 8 the sums $\sum_{Im(\rho)>0} R(x^{\rho})$ and

$\sum_{Im(\rho)>0} R(x^{1-\rho})$ - and likewise in equation 5 the sums $\sum_{Im(\rho)>0} Li(x^{\rho})$ and $\sum_{Im(\rho)>0} Li(x^{1-\rho})$ for $J(x)$ - need to be combined so as to pair terms corresponding to ρ and $1-\rho$ in order to make sense of these formulae; and even then the combined sum in each case is only conditionally convergent within the classical convergence framework.

It would be interesting instead to consider these sums *independently* as geometric generalised Césaro sums, taking account of the locations of the NT-roots ρ and $1-\rho$ (cf our earlier work in [VII] - [X]); and also to consider how the reversal of order of summation undertaken to pass formally from functions $Li(x^{\frac{\rho}{n}})$ to $R(x^{\frac{\rho}{n}})$ plays out in this context when treated more carefully via generalised Césaro arrays (see [IV] - [VI]).

It is similarly interesting to reconsider the Gram's series 10 in light of our discussion of TLA-coefficient functions and the interpretation of their extension to the left for $k \in \mathbb{Z}_{\leq 0}$.

Example (3) [$\psi(x)$]: Finally, by mirroring what was done for $J(x)$ but instead working with the function $\psi(x) := \sum_{p^n<x} \log p = \sum_{n<x} \Lambda(n)$ and its associated Stieltjes measure $\mathrm{d}\psi(x) = (\log x)\,\mathrm{d}J(x)$, we can obtain an alternative formula analogous to equation 1, namely that

$$\begin{aligned} \frac{\zeta'(s)}{\zeta(s)} &= \frac{d}{ds}\left(\log(\zeta(s))\right) = \frac{d}{ds}\int_0^\infty x^{-s}\mathrm{d}J(x) = -\int_0^\infty x^{-s}\log(x)\mathrm{d}J(x) \\ &= -\int_0^\infty x^{-s}\mathrm{d}\psi(x) = -s\int_0^\infty \psi(x)x^{-s-1}\,\mathrm{d}x = -s\cdot\mathcal{M}[\psi](-s) \quad (11) \end{aligned}$$

This is in fact cleaner to work with than equation 1 because $\frac{\zeta'(s)}{\zeta(s)}$ has simple poles rather than log-singularities and branch cuts (as occur for $\log(\zeta(s))$ at all the generalized roots of zeta (i.e. at its trivial and non-trivial roots and at its pole at $s=1$).

This makes the Mellin inversion to obtain $\psi(x)$ as $\frac{1}{2\pi i}\cdot\int_{a-i\infty}^{a+i\infty}\left[-\frac{\zeta'(s)}{\zeta(s)}\right]x^s\frac{\mathrm{d}s}{s}$ more tractable on combining with Hadamard's product formula for ζ, and leads instead to von Mangoldt's famous formula as a result analogous to equation 5, namely that for $x>1$

$$\psi(x) = x - \sum_{\rho\in NT}\frac{x^\rho}{\rho} + \sum_{m=1}^{\infty}\frac{x^{-2m}}{2m} - \frac{\zeta'(0)}{\zeta(0)} \quad . \tag{12}$$

Comment [Current understanding and extension]: Once again, while this is a long-established result, making sense of the sum $\sum_{\rho\in NT}\frac{x^\rho}{\rho}$ relies on pairing non-trivial roots ρ and $1-\rho$ and is even then only conditionally convergent classically. As such, re-examining it as two independent sums, $\sum_{Im(\rho)>0}$ and $\sum_{Im(\rho)<0}$, within the generalised geometric Césaro context would be interesting; and it would likewise be interesting to attempt an alternative derivation of equation 12 from equation 11 by inverting the Mellin transform not using the line integral $\int_{a-i\infty}^{a+i\infty}$ as done by von Mangoldt, but rather by working with the associated TLA-coefficient function for $\psi(x)$ and invoking the sort of arguments and results developed along these lines in [XI] - [XIII]. We go part way towards this Taylor-series-to-the-left re-derivation for $\psi(x)$ in section 2.4.

With these three examples as inspiration (or perhaps more as aspiration), let us turn briefly to considering these Taylor-series-to-the-left methods - and specifically our results from [XI]-[XIII] for Mellin transforms and TLA-coefficient functions within the generalised Césaro framework - in the context of their application to p-sum functions.

The discussion which follows in the rest of this section will initially consider in detail much simpler cases than those canvassed in examples (1)-(3) above, but these will enable us to cleanly demonstrate the intimate interaction between generalised Césaro convergence results and the behaviour of p-sum functions and their Mellin transforms.

This in turn will allow us to discuss in sharpened form some of the questions raised in the discussion of (1)-(3) above. Indeed in section 2.4 we will return explicitly to consider the case of $\psi(x)$ and also the new case of $\sigma_{\mu}(x) := \sum_{j<x} \mu(j)$ and the closely-related question of p-sum functions of strongly-periodic sequences.

Hopefully all of this will convince the reader of the value of reconsidering p-sum functions and their Mellin transforms using Taylor-series-to-the-left methods and the results of [XI]-[XIII]; and in turn, of re-assessing many of the key sums and formulae in the existing theory of the Riemann zeta function using tools - such as geometric generalised Césaro summation, Césaro arrays, and TLA-coefficient functions extended both left and right - which we developed in our earlier sets of papers on generalised Césaro convergence and in the previous papers in this set.

2.2 Mellin transforms and TLA-coefficient functions for p-sum functions in general

Suppose $\sigma(x)$ is a p-sum function on $[0,\infty)$. By this we mean a step-function with finite steps of height a_n at a distinct, positive, increasing sequence of points $\{x_n\}_{n=1}^{\infty}$, with $x_n \to \infty$ as $n \to \infty$; so that $\sigma(x)$ can be written as

$$\sigma(x) = \sum_{x_n<x} a_n \quad . \tag{13}$$

Here, and in all that follows, this characterisation holds exactly for generic $x \in [0,\infty) \setminus \{x_n\}_{n=1}^{\infty}$; and when $x = x_N$ coincides with a step-point we mean that $\sigma(x_N) = \frac{1}{2}\lim_{\epsilon\to 0}\left[\sigma(x_N-\epsilon)+\sigma(x_N+\epsilon)\right] = \sum_{j=1}^{N-1} a_j + \frac{1}{2}a_N$ so as to "split the difference" in the usual way.

Then $\sigma(x)$ is identically zero in a neighbourhood of 0 and if the growth of $\sigma(x)$ is sufficiently controlled as $x \to \infty$, e.g. $\sigma(x) = O(x^{\beta})$ for some $\beta \in \mathbb{R}$, its Mellin transform

$$\mathcal{M}[\sigma](s) := \int_0^{\infty} \sigma(x) \cdot x^{s-1}\,\mathrm{d}x \tag{14}$$

is classically well-defined on the half-plane $Re(s) < -\beta$ and can be trivially extended to all of $\mathbb{C}$ (with possible singularities) under the generalised Césaro convergence framework.

For this natural Césaro interpretation we consider $\int_0^X \sigma(x) \cdot x^{s-1}\,\mathrm{d}x$. Using integration by parts we have that

$$\begin{aligned} \int_0^X \sigma(x) \cdot x^{s-1}\,\mathrm{d}x &= \left[\frac{x^s}{s} \cdot \sigma(x)\right]_0^X - \frac{1}{s}\int_0^X \sum_j a_j \delta_{x_j}(x) \cdot x^s\,\mathrm{d}x \\ &= \frac{1}{s} X^s \sigma(X) - \frac{1}{s} \sum_{x_j < X} a_j x_j^s \end{aligned} \tag{15}$$

and $\mathcal{M}[\sigma](s)$ is defined as the generalised Césaro limit as $X \to \infty$ of this expression.

As derived in [XI]-[XIII], the TLA-coefficient function of σ is then given by

$$\overset{\vee}{\sigma}(s) = -\mathcal{M}[\sigma](-s) \cdot \frac{\sin(2\pi s)}{2\pi} \quad . \tag{16}$$

Since σ is identically zero in a neighbourhood of 0, the main result from section 2.3 of [XIII] (which relates the integer values of $\overset{\vee}{\sigma}(s)$ simultaneously to the coefficients of the power series for σ near 0 and near ∞), then tells us that in this case the values $\left\{\overset{\vee}{\sigma}(m)\right\}_{m \in \mathbb{Z}}$ give the coefficients of the power series describing $\sigma(x)$ as $x \to \infty$. Specifically,

$$\sigma(x) = -\sum_{m=-\infty}^{\infty} \overset{\vee}{\sigma}(m) x^m \qquad \text{as} \quad x \to \infty \tag{17}$$

where we expect $\overset{\vee}{\sigma}(m)$ to be identically zero for all sufficiently large m.

In fact, the result we have relied on here from [XIII] pre-supposes that the function being considered, $f(x)$, has clean power series involving only integer powers of x for x near 0 and near ∞ and a Mellin transform $\mathcal{M}[f](-s)$ with no poles at any $s \in \mathbb{C} \setminus \mathbb{Z}$. This ensures that its only Mellin transform poles lie at points $m \in \mathbb{Z}$ where $\overset{\vee}{\sigma}(m)$ picks up the associated residue, and this residue represents the difference between the coefficients of x^m in the power series for f around $x = 0$ and $x = \infty$ respectively.

If, more generally, $\mathcal{M}[f](-s)$ also has a pole at $s = \rho \notin \mathbb{Z}$ with residue c_ρ, then this will still correspond to the presence of a term x^ρ in the power series for $f(x)$ either around 0 or around ∞ or around both; and if f is a p-sum function then this term $c_\rho x^\rho$ will be attributable entirely to the asymptotic expansion as $x \to \infty$.

However, in such cases, the TLA-coefficient function no longer isolates this, since $\frac{\sin(2\pi s)}{2\pi}$ no longer has a zero at $s = \rho$ to cancel off the pole of $\mathcal{M}[f](-s)$ there and leave the residue c_ρ. In such cases it is therefore easier to work with the Mellin transform and its poles and residues directly, rather than passing to the TLA-coefficient function.

Bearing in mind this extension, let us now consider the Mellin transforms of some example p-sum functions in detail, starting from the simplest of cases in order to see in detail how the generalised Césaro character of such calculations manifests itself and how it relates to the power series asymptotics of the p-sum function itself.

2.3 Examples illustrating the generalised Césaro characteristics of Mellin transforms of p-sum functions and what these imply for their behaviour as $x \to \infty$

Case (i): Let $\sigma_0(x)$ be the simplest of all p-sum functions $\sigma_0(x) := \sum_{j<x} 1$. Then equation 15 gives that

$$\int_0^X \sigma_0(x) \cdot x^{s-1}\,\mathrm{d}x = \frac{1}{s} X^s (X-\alpha) - \frac{1}{s} \cdot \sum_{j=1}^{k} j^s$$

on setting $x_j = j$ and $a_j = 1$ for all j, and writing $X = k + \alpha$ in the usual way.

Now recall from [I]-[III] that

$$\sum_{j=1}^{k} j^s = \frac{X^{s+1}}{s+1} + \zeta(-s) - R(X;s) \tag{18}$$

where $R(X;s)$ is strongly Césaro-asymptotic to zero as $X \to \infty$ via the pure power of the Césaro operator, $P^{Floor(Re(s))+1}$. Recall likewise that both X^{s+1} and $X^s\alpha$ are also Césaro-asymptotic to zero (see [III]). It follows that we have

$$\int_0^X \sigma_0(x) \cdot x^{s-1}\,\mathrm{d}x \overset{C}{\sim} -\frac{\zeta(-s)}{s} \tag{19}$$

and taking Césaro limits we have that

$$\mathcal{M}[\sigma_0](s) = -\frac{\zeta(-s)}{s} \qquad \text{and therefore} \qquad -\mathcal{M}[\sigma_0](-s) = -\frac{\zeta(s)}{s} \,. \tag{20}$$

Now consider this formula. Since $\zeta(s)$ has a single pole at $s = 1$ with residue 1, so $-\mathcal{M}[\sigma_0](-s)$ has just two poles - one at $s = 1$ with residue -1 and one at $s = 0$ with residue $-\zeta(0) = \frac{1}{2}$. The TLA-coefficient function of $\sigma_0(x)$ is $\overset{\vee}{\sigma}_0(s) = -\frac{\zeta(s)}{s} \cdot \frac{\sin(2\pi s)}{2\pi}$ and it follows that for integer m this has

$$\overset{\vee}{\sigma}_0(m) = \begin{cases} \frac{1}{2} & , \quad m = 0 \\ -1 & , \quad m = 1 \\ 0 & , \quad \mathit{else.} \end{cases} \tag{21}$$

Since σ_0 is identically zero in a neighbourhood of 0 it follows finally per equation 17 that we should have

$$\sigma_0(x) = x - \frac{1}{2} \qquad \text{as} \qquad x \to \infty \,. \tag{22}$$

In what sense is this true? The answer is that, by equation 18 when $s = 0$, it is true in precisely a *strong* generalised Césaro sense, i.e. $\sigma_0(x) \overset{C}{\simeq} x - \frac{1}{2}$ meaning that

$$\sigma_0(x) = x - \frac{1}{2} + R_0(x) \qquad \text{where} \qquad R_0(x) \overset{C}{\simeq} 0 \tag{23}$$

as $x \to \infty$ via a pure power of P.

This is reasonable in light of our observation in [XIII] that the generalised geometric Césaro convergence framework is the right setting for all analysis of Mellin transforms and TLA-coefficient functions, and for all integration and power series analysis of functions around 0 and ∞; and given that strong Césaro convergence to zero may still entail residual complex oscillatory behaviour that does not correspond to any explicit power-divergence.

Indeed, since the poles and residues at the points $s = 0$ and $s = 1$ alone give rise to the power components $x - \frac{1}{2}$ in this expansion, we surmise that this complex oscillatory residual term $R_0(X)$ in some sense represents a superposition of powers x^s from all the remaining continuum of s-values where $\mathcal{M}[\sigma_0](-s)$ is non-singular, although how this hypothetical superposition might actually be understood concretely in detail is unclear.

Does this way of interpreting things continue to hold true for other p-sum functions?

Case (ii): Let us consider next the p-sum function $\sigma_1(x) := \sum_{j<x} j$. By equation 15 we have, in this case, that

$$\int_0^X \sigma_1(x) \cdot x^{s-1}\,\mathrm{d}x = \frac{1}{s} X^s \cdot \left\{ \frac{1}{2}(X-\alpha)^2 + \frac{1}{2}(X-\alpha) \right\} - \frac{1}{s} \cdot \sum_{j=1}^{k} j^{s+1}$$

and from equation 18 we have that

$$\sum_{j=1}^{k} j^{s+1} = \frac{X^{s+2}}{s+2} + \zeta(-s-1) - R(X;s)\,.$$

Here $R(X;s) \overset{C}{\simeq} 0$ as $X \to \infty$ via $P^{Floor(Re(s))+2}$, while by [III] we know that all of X^{s+2}, X^{s+1}, $X^{s+1}\alpha$, $X^s\alpha^2$ and $X^s\alpha$ are also Césaro convergent to zero. It follows that

$$\int_0^X \sigma_1(x) \cdot x^{s-1}\,\mathrm{d}x \overset{C}{\sim} -\frac{\zeta(-s-1)}{s} \tag{24}$$

and taking Césaro limits we have that

$$\mathcal{M}[\sigma_1](s) = -\frac{\zeta(-s-1)}{s} \qquad \text{and therefore} \qquad -\mathcal{M}[\sigma_1](-s) = -\frac{\zeta(s-1)}{s}\,. \tag{25}$$

Thus $-\mathcal{M}[\sigma_1](-s)$ again has two poles - one at $s = 0$ with residue $-\zeta(-1) = \frac{1}{12}$, the other at $s = 2$ with residue $-\frac{1}{2}$. And thus for integer m the TLA-coefficient function of σ_1, namely $\overset{\vee}{\sigma}_1(s) = -\frac{\zeta(s-1)}{s} \cdot \frac{\sin(2\pi s)}{2\pi}$, satisfies

$$\overset{\vee}{\sigma}_1(m) = \begin{cases} \frac{1}{12} & , \quad m = 0 \\ -\frac{1}{2} & , \quad m = 2 \\ 0 & , \quad \textit{else.} \end{cases} \tag{26}$$

Since σ_1 is identically zero in a neighbourhood of 0 we conclude, either from equation 17 applied to $\overset{\vee}{\sigma}_1(s)$ or directly from the poles and residues of $-\mathcal{M}[\sigma_1](-s)$, that $\sigma_1(x)$ has the asymptotic power series expansion

$$\sigma_1(x) = \frac{1}{2}x^2 - \frac{1}{12} \qquad \text{as} \qquad x \to \infty \ . \tag{27}$$

As in case (i), this is again true exactly in a strong generalised Césaro asymptotic sense, since equation 18 with $s = 1$ says precisely that

$$\sigma_1(x) = \frac{1}{2}x^2 - \frac{1}{12} - R_1(X) \qquad \text{where} \qquad R_1(X) \overset{C}{\simeq} 0 \tag{28}$$

via P^2.

In other words, for this p-sum function also, the poles and residues of its Mellin transform (or alternatively the integer values of its TLA-coefficient function) give the correct asymptotic power series expansion for it as $x \to \infty$, as long as this is understood in a generalised geometric Césaro sense - i.e. as holding modulo a remainder function which is *strongly* Césaro asymptotic to zero (and which can perhaps be understood as representing some sort of superposition arising from all the non-singular values of the Mellin transform at $s \in \mathbb{C} \backslash \{0, 2\}$).

Case (iii): Finally, consider the general case here of $\sigma_\nu(x) := \sum_{j<x} j^\nu$, $\nu \in \mathbb{C}$. Then the logic applied in cases (i) and (ii) works in identical fashion and we end up with

$$\int_0^X \sigma_\nu(x) \cdot x^{s-1}\,\mathrm{d}x \ \overset{C}{\sim} \ -\frac{\zeta(-s-\nu)}{s} \ . \tag{29}$$

Thus

$$\mathcal{M}[\sigma_\nu](s) = -\frac{\zeta(-s-\nu)}{s} \qquad \text{and therefore} \qquad -\mathcal{M}[\sigma_\nu](-s) = -\frac{\zeta(s-\nu)}{s} \tag{30}$$

and $-\mathcal{M}[\sigma_\nu](-s)$ again has precisely two poles - one at $s = 0$ with residue $-\zeta(-\nu)$, the other at $s = \nu + 1$ with residue $-\frac{1}{\nu+1}$.

Since generically $\nu + 1 \notin \mathbb{Z}$, passing to the TLA-coefficient function in this case does not allow direct invocation of equation 17 and so is not productive. However, as discussed earlier and as seen in cases (i) and (ii) we can still simply work directly from the poles and residues of $-\mathcal{M}[\sigma_\nu](-s)$ to conclude that $\sigma_\nu(x)$ has the asymptotic expansion

$$\sigma_\nu(x) = \frac{x^{\nu+1}}{\nu+1} + \zeta(-\nu) \qquad \text{as} \qquad x \to \infty \ . \tag{31}$$

This again reflects exactly the generalised Césaro asymptotic nature of $\sigma_\nu(x)$, since by equation 18 with $s = \nu$ we know that

$$\sigma_\nu(x) = \frac{x^{\nu+1}}{\nu+1} + \zeta(-\nu) - R_\nu(X) \qquad \text{where} \qquad R_\nu(X) \overset{C}{\simeq} 0 \tag{32}$$

via $P^{Floor(Re(\nu))+1}$.

Thus the poles and residues of the Mellin transform of σ_ν do give the correct asymptotic power series expansion for $\sigma_\nu(x)$ as $x \to \infty$ as long as we always understand this modulo a residual remainder function which is strongly Césaro-asymptotic to zero (and which may be interpretable as some form of mysterious superposition of powers x^s over the continuum of s-values outside this set of poles).

2.4 Revisiting the Key p-sum functions from the theory of ζ

Having arrived at this general understanding, let us reconsider the key p-sum functions considered in examples (1)-(3) in section 2.1, in particular the case of $\psi(x)$ in (3).

Revisiting example (3) [$\psi(x)$]: The classical derivation of von Mangoldt's formula

$$\psi(x) = x - \sum_{\rho \in NT} \frac{x^\rho}{\rho} + \sum_{m=1}^{\infty} \frac{x^{-2m}}{2m} - \frac{\zeta'(0)}{\zeta(0)} \tag{33}$$

involves extensive complex integration and careful estimation. Can it be approached instead via the treatment of Mellin transforms of p-sum functions developed in this section?

Well, by equation 11 we know that the Mellin transform of $\psi(x)$ is given by

$$-\mathcal{M}[\psi](-s) = \frac{\zeta'(s)}{s\zeta(s)} \ . \tag{34}$$

Now $\frac{\zeta'(s)}{\zeta(s)}$ has simple poles precisely at the roots and pole of zeta, each with residue 1 if a root or -1 if a pole. So the poles of $-\mathcal{M}[\psi](-s)$ occur at (a) the pole of zeta at $s = 1$ with residue -1, (b) the trivial roots of zeta at $s = -2m$ ($m \in \mathbb{Z}_{>0}$) with residues $-\frac{1}{2m}$, (c) the NT-roots of zeta $s = \rho$ with residues $\frac{1}{\rho}$ and (d) the point $s = 0$ with residue $\frac{\zeta'(0)}{\zeta(0)}$.

But then it follows at once by the theory we have developed in this section that $\psi(x)$ *should* have the asymptotic expansion

$$\psi(x) = x - \sum_{\rho \in NT} \frac{x^\rho}{\rho} + \sum_{m=1}^{\infty} \frac{x^{-2m}}{2m} - \frac{\zeta'(0)}{\zeta(0)} \qquad \text{as} \qquad x \to \infty \quad . \tag{35}$$

We see that our theory, using the results from [XI]-[XIII] and their application to p-sum functions here, in fact recovers the full von Mangoldt formula immediately - and seemingly with almost no effort - at least as an asymptotic result.

It is worth, however, parsing the differences in these two formulations. Von Mangoldt's result is an exact result provided that the sum over NT-roots is taken with the roots at ρ and $1-\rho$ paired in advance and with the combined sum then taken under conditional convergence over all $Im(\rho) > 0$.

By contrast, our theory tells us that formula 35 is true as a *strong* Césaro asymptotic statement, i.e. modulo a residual remainder function $R_\psi(x)$ which satisfies that $P^N[R_\psi](x)$ converges classically to zero as $x \to \infty$ for some sufficiently high power, N, of the Césaro averaging operator.

Comment: Understanding the interaction between these two formulations would be very interesting. In particular, how does the process of pairing roots in the classical approach impact the formation of the components of $R_\psi(x)$ - which in the Césaro framework would instead consist of two independent geometric sums over $Im(\rho) > 0$ and $Im(\rho) < 0$, each taking account of the location of the NT-roots. In particular how does it lead to the partial cancellation of these components when paired, so as to leave an exact formula, at least in a conditionally convergent sense?

Alternatively, it would likewise be very interesting to consider the RHS of equation 35 directly in a generalised Césaro sense by expanding each power of x as $x^\nu = \sum_{n=0}^{\infty} \frac{\nu^n (\log x)^n}{n!}$ and using the methodology of Césaro arrays developed in [IV]-[VI] to reverse the order of the summations over roots of ζ and over n.

In doing so it is tempting to imagine, based on the generalised root identities for ζ for $\mu \in \mathbb{Z}_{\leq 1}$ and $s_0 = 0$, that all the component-1 pieces at each "height" should be identically zero, leaving us only to re-assemble the component-2 pieces arising from non-trivial Césaro eigenfunction divergences at these heights. However, this would be a mistake.

In this case we wish to add in each power (x, x^ρ or x^{-2m}) geometrically *at* the corresponding root (1, ρ or $-2m$) in the complex plane. By contrast, the root identities for ζ with $s_0 = 0$ entail adding in each term at the *negative* of the associated root in $\mathbb{C}$ (i.e. at -1, $-\rho$ or $2m$). As such, since the geometric location of summands is critical in geometric generalised Césaro theory, we cannot simply quote the root identities for ζ for this Césaro array computation.

Nonetheless, the corresponding computations for the case we are now interested in can be carried out in a fashion analogous to that undertaken in [VII]-[X] under the assumption of the Riemann hypothesis (RH).

When we do so we still obtain a simple, elegant formula for these component-1 pieces. Moreover, we find that the non-trivial Césaro eigenfunction divergences which form the component-2 pieces at each height *can* seemingly be re-combined in a concise way as a pair of generalised Césaro integrals over the positive and negative halves of the critical line - specifically as integrals of the function $\frac{x^w}{w}$ against the measures given, respectively, by $\frac{\mathrm{d}}{\mathrm{d}w}\left(\ln\left(\frac{\mathrm{e}^{\mp\frac{i\pi}{2}w}\Gamma(w)}{(2\pi)^w}\right)\right)\mathrm{d}w$.

These Césaro array developments seem to us[4] both extremely interesting and extremely promising. We have not, however, quite succeeded thus far in

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[4] the author and Otis Yeere (pers. comm.)

overcoming a number of residual calculational anomalies and thereby arriving at a final new formula for $\psi(x)$ (modulo RH) which could be investigated numerically for correctness. We hope to take up this calculation in a future paper.

Briefly revisiting examples (1) and (2) [$J(x)$ and $\pi(x)$]: In a similar fashion, it is also interesting to apply this TS-to-the-left perspective to the other two key p-sum functions, $J(x)$ and $\pi(x)$, from (1) and (2). In these cases the presence of log-singularities in the Mellin transforms leads to terms of the form $Li(x)$, $Li(x^\rho)$ and $Li(x^{-2m})$, rather than simply powers (x, x^ρ, x^{-2m}). But the question of relating, on the one hand, the pairing of NT-roots and the conditional convergence of resulting sums to, on the other, the presence of residual remainder pieces strongly Césaro asymptotic to zero under our TS-to-the-left perspective, remains of great interest. As does the possibility of expanding $Li(x)$ as an asymptotic series in $\frac{x}{(\log x)^n}$ and then using Césaro arrays to tackle the reversal of order of summation over roots and over n with a view to potentially identifying new expressions for $J(x)$ and $\pi(x)$.

A new example (4) [$\sigma_\mu(x)$]: We conclude our consideration of p-sums in the theory of ζ with one further important example. Let $\sigma_\mu(x) := \sum_{j<x} \mu(j)$ where μ is the well-known Möbius function. Then

$$\int_0^X \sigma_\mu(x) \cdot x^{s-1}\,\mathrm{d}x = \frac{1}{s} X^s \sigma_\mu(X) - \frac{1}{s} \sum_{j<X} \mu(j) j^s \quad .$$

Now in this case we do not have a generalised Césaro expression either for $\sigma_\mu(X)$ or for $\sum_{j<X} \mu(j)j^s$, but it is clear that for $Re(s) < -1$, $\frac{1}{s}X^s\sigma_\mu(X) \to 0$ classically as $X \to \infty$. And since for such s we also have that $\frac{1}{s}\sum_{j<X}\mu(j)j^s \to -\frac{1}{s}\frac{1}{\zeta(-s)}$ by Möbius inversion, it follows that we have

$$-\mathcal{M}[\sigma_\mu](-s) = -\frac{1}{s}\frac{1}{\zeta(s)} \ . \tag{36}$$

This holds initially for $Re(s) < -1$ but then extends by analytic continuation to hold as a generalised Césaro result for all $s \in \mathbb{C}$.

Now $-\frac{1}{s}\frac{1}{\zeta(s)}$ has poles at (a) $s = -2m$ with residues $\frac{1}{2m\zeta'(-2m)}$ and at (b) $s = \rho \in NT$ with residues $\frac{-1}{\rho\zeta'(\rho)}$ and (c) at $s = 0$ with residue $\frac{-1}{\zeta(0)} = 2$, but (unlike for the Mellin transform of ψ) no pole at $s = 1$.

We therefore hypothesize that in a strong Césaro sense we should have

$$\sigma_\mu(x) \overset{C}{\simeq} \sum_{\rho\in NT} \frac{x^\rho}{\rho\zeta'(\rho)} - \sum_{m=1}^{\infty} \frac{x^{-2m}}{2m\zeta'(-2m)} - 2 \qquad \text{as} \qquad x \to \infty \ . \tag{37}$$

The fact that there is no pole at $s = 1$ means that $\sigma_\mu(x)$ has no x^1-divergence, unlike $\psi(x)$. Its most "divergent" pieces are those arising from the poles at NT-roots ρ, which have an oscillatory character and are thus subject to much cancellation - in keeping with the fact that the terms $\mu(j)$ in the p-sum for $\sigma_\mu(x)$ are ± 1, so that $\sigma_\mu(x)$ is quasi-alternating.

Note that in writing equation 37 for $\sigma_\mu(x)$ we have assumed that none of the expressions $\zeta'(\rho)$ are zero, or equivalently that all the NT-roots of zeta are simple roots of order 1. If any of them are in fact double-roots or roots of higher order, then these would immediately correspond in equation 36 to poles of order 2 or higher in $-\mathcal{M}[\sigma_\mu](-s)$ and $\sigma_\mu(x)$ would then instead gain power-log terms of the form $x^\rho(\log x)^m$ for some $m \in \mathbb{Z}_{>0}$ in equation 37.

Example (v): We conclude this series of examples and this section with a very brief discussion of a separate interesting avenue of exploration regarding p-sum functions, their Mellin transforms and their asymptotics as $x \to \infty$.

In [V] we considered in detail series $\sum_{j=1}^{\infty} a_j$ as the discrete generalised Césaro limits of p-sum sequences $\sigma_k := \sum_{j=1}^{k} a_j$. Where these are "strongly periodic" we were able to deduce a number of results about the *discrete* generalised Césaro values of sums of the form $\sum_{j=1}^{\infty} a_j j^s$ and hence derive by generalised Césaro means several well-known theorems regarding exponential sums.

Now the expression $\sum_{j=1}^{\infty} a_j j^s$ would turn up naturally in the limit as $X \to \infty$ from applying equation 15 to the p-sum function $\sigma(x) := \sum_{j=1}^{k} a_j$. Since this is the continuous version of the p-sum sequence $\sigma_k := \sum_{j=1}^{k} a_j$ which we started with, it is natural to consider the Mellin transform of such a p-sum function $\sigma(x)$; and also whether there are any resulting implications regarding its asymptotic behaviour as $x \to \infty$ which might arise from understanding the poles and residues of $-\mathcal{M}[\sigma](-s)$.

In doing so, however, it would be important to note that all our discussion of Mellin transforms of p-sum functions in this section has been undertaken within the *continuous* generalised Césaro framework. In [V], by contrast, most of the working and results were obtained within the *discrete* generalised Césaro framework. In subsection 3.3 in [V] we noted some of the differences in analysis which would arise were we to move from the discrete to the continuous Césaro framework, and these would of course have to be incorporated in calculating and interpreting our Mellin transforms.

We shall not delve any further or explore any such examples in detail in this paper, but will again hope to return to such considerations in a future paper. But we do consider it an interesting avenue to explore further and we wholeheartedly encourage readers to also jump on board this train and see where it might lead. Should there be any readers so moved, we would be delighted to collaborate with them in this venture - and we suggest meeting at Marwar junction and proceeding to the Degumber states to hatch plans[5].

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[5] Alternatively, simply contacting the author to propose such a collaboration might work equally well.

3 Miscellaneous further Taylor-series-to-the-left results for integration

In this section we consider a miscellany of new integration results obtainable by Taylor-series-to-the-left methods - results which are either beautiful in their own right or where alternative approaches are hard to identify. We start with an example showing that Taylor-series-to-the-left methodology can sometimes provide a more direct alternative to the use of Feynman's trick.

Example Set 1 [$\int_0^\infty \mathrm{e}^{-x^2} \cdot \cos(\alpha x)\, \mathrm{d}x$]: Integrals like $\int_0^\infty \mathrm{e}^{-x^2} \cdot \cos(3x)\, \mathrm{d}x$ or $\int_0^\infty \mathrm{e}^{-x^2} \cdot \cos(4x)\, \mathrm{d}x$ are traditionally handled by Feynman's trick - to wit: introduce a parameter α and consider $\int_0^\infty \mathrm{e}^{-x^2} \cdot \cos(\alpha x)\, \mathrm{d}x$ as a function $f(\alpha)$; find a differential equation for f (in this case that $f'(\alpha) = -\frac{\alpha}{2} f(\alpha)$) by differentiating under the integral w.r.t. α and using integration by parts; solve to get $f(\alpha) = C\mathrm{e}^{-\frac{\alpha^2}{4}}$ and calculate C from the known value in the case of $\alpha = 0$ to obtain $f(\alpha) = \frac{\sqrt{\pi}}{2}\mathrm{e}^{-\frac{\alpha^2}{4}}$; hence evaluate the cases of interest as $f(3)$ or $f(4)$ etc.

Instead we can attack this directly using our methods from [XI]-[XIII], in particular lemma 1a from [XII] regarding TLA-coefficient functions of products.

Let $f(x) = \mathrm{e}^{-x^2}$ and $g(x) = \cos(\alpha x)$ so that $h(x) := f(x) \cdot g(x)$ is our integrand. Then we have

$$\overset{\vee}{f}(s) = \cos(\pi s) \cdot \cos(\frac{\pi s}{2}) \cdot \frac{1}{(\frac{s}{2})!} \qquad \text{and} \qquad \overset{\vee}{g}(s) = \cos(\pi s) \cdot \cos(\frac{\pi s}{2}) \cdot \frac{\alpha^s}{s!}$$

and in both cases, the integer-values of these TLA-coefficient functions arise entirely from their f_0 and g_0 components, i.e. $f_\infty(m) \equiv 0$ and $g_\infty(m) \equiv 0$, so that $f_0(m) = \overset{\vee}{f}(m)$ and $g_0(m) = \overset{\vee}{g}(m)$ for all $m \in \mathbb{Z}$. It follows by lemma 1a in [II] that we have

$$\overset{\vee}{h}(s) \;=\; \sum_{j=-\infty}^{\infty} g_0(j) \cdot f_0(s-j) \;=\; \sum_{j=-\infty}^{\infty} \overset{\vee}{g}(j) \cdot \overset{\vee}{f}(s-j)$$

and on noting that $\overset{\vee}{g}(j) = 0$ for all $j \in \mathbb{Z}_{<0}$ and for all $j \in \mathbb{Z}_{\geq 0}$ odd, and so writing $j = 2l$; and after noting that $\cos(\pi(s-2l)) = \cos(\pi s)$ and $\cos(\frac{\pi}{2}(s - 2l)) = (-1)^l \cos(\frac{\pi}{2}s)$, we have that

$$\begin{aligned} \overset{\vee}{h}(s) \;&=\; \sum_{l=0}^{\infty} (-1)^l \, \frac{\alpha^{2l}}{(2l)!} \cdot \cos(\pi s) \cdot (-1)^l \cos(\frac{\pi}{2}s) \cdot \frac{1}{(\frac{s}{2} - l)!} \\ &=\; \cos(\pi s) \cdot \cos(\frac{\pi}{2}s) \cdot \sum_{l=0}^{\infty} \frac{\alpha^{2l}}{(2l)!} \cdot \frac{1}{(\frac{s}{2} - l)!} \qquad . \end{aligned}$$

It follows at once that $\overset{\vee}{h}(-1) = 0$ and since $\cos(\pi(-1 + \epsilon)) = -1 + O(\epsilon^2)$,

$\cos(\frac{\pi}{2}(-1+\epsilon)) = \frac{\pi\epsilon}{2}\epsilon + O(\epsilon^3)$ and $\left(\frac{-1}{2}\right)! = \sqrt{\pi}$, so we also have that

$$\begin{aligned}
\overset{\vee}{h}'(-1) &= \lim_{\epsilon\to 0}\frac{\overset{\vee}{h}(-1+\epsilon)}{\epsilon} = -\frac{\pi}{2}\cdot\sum_{l=0}^{\infty}\frac{\alpha^{2l}}{(2l)!}\cdot\frac{1}{(\frac{-1}{2}-l)!} \\
&= -\frac{\pi}{2}\cdot\sum_{l=0}^{\infty}\frac{\alpha^{2l}}{(2l)!}\cdot\frac{(\frac{-1}{2})\cdot(\frac{-3}{2})\cdots(\frac{-1}{2}-(l-1))}{\sqrt{\pi}} \\
&= -\frac{\sqrt{\pi}}{2}\cdot\sum_{l=0}^{\infty}\frac{\alpha^{2l}}{(2l)!}\cdot(-1)^l\cdot\frac{1\cdot 3\cdot 5\cdots(2l-1)}{2^l} \\
&= -\frac{\sqrt{\pi}}{2}\cdot\sum_{l=0}^{\infty}\frac{\alpha^{2l}}{2^l\cdot l!}\cdot\frac{(-1)^l}{2^l} \\
&= -\frac{\sqrt{\pi}}{2}\cdot\sum_{l=0}^{\infty}(-1)^l\cdot\frac{\left(\frac{\alpha^2}{4}\right)^l}{l!} = -\frac{\sqrt{\pi}}{2}\cdot\mathrm{e}^{-\frac{\alpha^2}{4}} \quad .
\end{aligned}$$

Hence finally $\int_0^\infty \mathrm{e}^{-x^2}\cdot\cos(\alpha x)\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}\cdot\mathrm{e}^{-\frac{\alpha^2}{4}}$ and we have obtained the same answer as previously derived using Feynman's trick.

Comment: The artistry of Feynman's trick relies on the integrand being of such a form that the parametric function $f(\alpha)$ ends up satisfying a tractable differential equation on differentiating under the integral.

In the Taylor-series-to-the-left approach above this is reflected instead in the final formula for $\overset{\vee}{h}'(-1)$ being susceptible to algebraic simplification.

Note, however, that in cases where Feynman's trick encounters difficulties and does not become tractable, the Taylor-series-to-the-left approach may still generate a closed-form formula (albeit perhaps as an infinite sum) for the integral, which in turn may be useful either for numerical implementation or asymptotic analysis. Thus, aesthetics aside, there may be some relative advantage to the new Taylor-series-to-the-left methodology in such cases.

Example Set 2 [$\int_0^\infty \frac{\mathrm{e}^{-x^n}}{1+x^2}\,\mathrm{d}x$]: Using traditional techniques it is not obvious how to attempt computation of $\int_0^\infty \frac{\mathrm{e}^{-x^n}}{1+x^2}\,\mathrm{d}x$ for any $n\in\mathbb{Z}_{>0}$.

Using lemma 1a from [XII] again, however, let $f(x) = \mathrm{e}^{-x^n}$ and $g(x) = \frac{1}{1+x^2}$, so that $h(x) := f(x)\cdot g(x)$ is our integrand. Then we know that

$$\overset{\vee}{f}(s) = \frac{1}{2n}\cdot\frac{\sin(2\pi s)}{\sin(\frac{\pi s}{n})}\cdot\frac{1}{(\frac{s}{n})!} \qquad \text{and} \qquad \overset{\vee}{g}(s) = \cos(\pi s)\cdot\cos(\frac{\pi s}{2})$$

and here all the integer-values of $\overset{\vee}{f}$ are attributable entirely to f_0 (i.e. $f_\infty(m)\equiv 0$ and $f_0(m) = \overset{\vee}{f}(m)$ for all $m\in\mathbb{Z}$); while for $\overset{\vee}{g}(m)$, the values with $m\in\mathbb{Z}_{\geq 0}$ are attributable to g_0 and the values with $m \in \mathbb{Z}_{<0}$ are attributable to g_∞, i.e.

$$g_0(m) = \begin{cases} \cos(\pi m) \cdot \cos(\frac{\pi m}{2}) & , \quad m \in \mathbb{Z}_{\geq 0} \\ 0 & , \quad m \in \mathbb{Z}_{<0} \end{cases}$$

and

$$g_\infty(m) = \begin{cases} 0 & , \quad m \in \mathbb{Z}_{\geq 0} \\ -\cos(\pi m) \cdot \cos(\frac{\pi m}{2}) & , \quad m \in \mathbb{Z}_{<0} \end{cases} \quad .$$

For $m \in \mathbb{Z}$ we then have

$$\overset{\vee}{h}(m) = \sum_{j=0}^{m} \overset{\vee}{f}(j) \cdot \overset{\vee}{g}(m-j) \quad . \tag{38}$$

Here the sum is only over non-negative j since $\overset{\vee}{f}(j) = 0$ for all $j \in \mathbb{Z}_{<0}$, and is only for $j \leq m$ because of the split above between g_0 and g_∞. Note also that here we have taken $\overset{\vee}{h}(m)$ in this form as a finite sum rather than as $\sum_{j=0}^{\infty} \overset{\vee}{f}(j) \cdot \overset{\vee}{g}(m-j)$ so that, on moving from $m \in \mathbb{Z}$ to arbitrary s and ultimately calculating $\overset{\vee}{h}'(-1)$, we end up with a convergent sum, rather than with a divergent sum requiring further technical analysis[6]).

To now move from $s = m \in \mathbb{Z}$ to arbitrary $s \in \mathbb{C}$ we then need simply to replace the finite sum in equation 38 by a difference of two remainder sums in the usual way (as discussed in examples in [XIII]), i.e. as

$$\overset{\vee}{h}(s) = R_{+,0}\left[\overset{\vee}{f}(z) \cdot \overset{\vee}{g}(s-z)\right](0) - R_{+}\left[\overset{\vee}{f}(z) \cdot \overset{\vee}{g}(s-z)\right](s) \quad . \tag{39}$$

It follows tautologically from this equation that $\overset{\vee}{h}(-1) = 0$. As usual we can thus obtain $\overset{\vee}{h}'(-1)$ as $\lim_{\epsilon \to 0} \frac{\overset{\vee}{h}(-1+\epsilon)}{\epsilon}$, and in equation 39 we have, after elementary simplifications, that

$$\overset{\vee}{h}(-1+\epsilon) = R_{+,0}\left[H(z;\epsilon)\right](0) - R_{+,0}\left[H(z;\epsilon)\right](\epsilon) \tag{40}$$

where

$$H(z;\epsilon) = \frac{1}{2n} \cdot \frac{\sin(2\pi z)}{\sin(\frac{\pi z}{n})} \cdot \frac{1}{(\frac{z}{n})!} \cdot \cos(\pi(1+z) - \pi\epsilon) \cdot \cos(\frac{\pi}{2}(1+z) - \frac{\pi}{2}\epsilon) \quad . \tag{41}$$

It is not, however, trivial to evaluate $\lim_{\epsilon \to 0} \frac{\overset{\vee}{h}(-1+\epsilon)}{\epsilon}$ using the expressions in equations 40 and 41 because there is ϵ-dependence not just in $H(z;\epsilon)$ but also in the geometric location of the summands in the second remainder sum.

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[6] There is a always such a choice in applying lemma 1a; it usually amounts to choosing between expressions which are convergent or asymptotic, and to consideration of s near 0 or near ∞

To proceed we therefore first work to $O(\epsilon^2)$ and split $H(z;\epsilon)$ explicitly as $H(z;\epsilon) = \tilde{H}_0(z) + \epsilon \cdot \tilde{H}_1(z) + O(\epsilon^2)$ so that the remainder sums can instead be taken on "fabrics" $\tilde{H}_i(z)$ which have no ϵ-dependence.

Since $\cos(\pi(1+z) - \pi\epsilon) = -\cos(\pi z) - \sin(\pi z) \cdot \pi\epsilon + O(\epsilon^2)$ and $\cos(\frac{\pi}{2}(1+z) - \frac{\pi}{2}\epsilon) = -\sin(\frac{\pi z}{2}) + \cos(\frac{\pi z}{2}) \cdot \frac{\pi}{2}\epsilon + O(\epsilon^2)$, we have

$$H(z;\epsilon) = \tilde{H}_0(z) + \epsilon \cdot \tilde{H}_1(z) + O(\epsilon^2) \tag{42}$$

where

$$\tilde{H}_0(z) = \frac{1}{2n} \cdot \frac{\sin(2\pi z)}{\sin(\frac{\pi z}{n})} \cdot \frac{1}{(\frac{z}{n})!} \cdot \cos(\pi z) \cdot \sin(\frac{\pi}{2} z) \tag{43}$$

and

$$\tilde{H}_1(z) = \frac{1}{2n} \cdot \frac{\sin(2\pi z)}{\sin(\frac{\pi z}{n})} \cdot \frac{1}{(\frac{z}{n})!} \cdot \pi \cdot \left\{ \begin{array}{c} -\frac{1}{2}\cos(\pi z) \cdot \cos(\frac{\pi}{2} z) \\ \\ +\sin(\pi z) \cdot \sin(\frac{\pi}{2} z) \end{array} \right\} \quad . \tag{44}$$

Now consider first the remainder sums in equation 40 applied to $\tilde{H}_0(z)$, namely

$$R_{+,0}\left[\tilde{H}_0(z)\right](0) - R_{+,0}\left[\tilde{H}_0(z)\right](\epsilon) \quad .$$

For any $j \in \mathbb{Z}_{\geq 0}$, in the second of these we have that

$$\tilde{H}_0(j+\epsilon) = \tilde{H}_0(j) + \epsilon \cdot \tilde{H}_0'(j)$$

and thus $R_{+,0}\left[\tilde{H}_0(z)\right](0) - R_{+,0}\left[\tilde{H}_0(z)\right](\epsilon)$ gives rise to two components.

The first is a p-sum function arising from adding in $\tilde{H}_0(j)$ at each $x = j \in \mathbb{Z}_{\geq 0}$ and then subtracting it off again at $x = j + \epsilon$, making it a p-sum function, $s(x)$, which is identically zero on most of $[0, \infty)$ except for up-and-down "steps" of height $\tilde{H}_0(j)$ and width ϵ at each point $x = j \in \mathbb{Z}_{\geq 0}$.

But, writing $X = k + \alpha$ as usual, it follows at once that $P[s](X) = \frac{1}{X} \cdot \epsilon \cdot \sum_{j=1}^{k} \tilde{H}_0(j)$ and since $\sum_{j=1}^{k} \tilde{H}_0(j)$ converges classically to a finite value as $X \to \infty$ it follows at once that this component contributes zero in the calculation of $\lim_{\epsilon \to 0} \frac{\overset{\vee}{h}(-1+\epsilon)}{\epsilon}$.[7]

The second component is simply $-\epsilon$ times the convergent sum $\sum_{j=1}^{\infty} \tilde{H}_0'(j)$, so that in $\lim_{\epsilon \to 0} \frac{\overset{\vee}{h}(-1+\epsilon)}{\epsilon}$ this immediately makes the contribution $-\sum_{j=1}^{\infty} \tilde{H}_0'(j)$.

As for the remainder sums applied to $\tilde{H}_1(z)$, exactly the same reasoning applies, giving a geometric Césaro component which is again zero and a second component consisting of $-\epsilon \cdot \sum_{j=1}^{\infty} \tilde{H}_1'(j)$. But since the $\tilde{H}_1(z)$ term in equation

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[7] This first component is what we have previously called the "geometric Césaro component" in the calculation of the derivative of a function defined by a remainder sum over a fabric. Where this remainder sum is not itself classically convergent, this geometric component need not be zero (as for example in our calculation of $\Gamma'(1)$ in [II]), but as here it will always be zero where the remainder sum is convergent.

42 is already multiplied by a factor of ϵ this leaves this contribution being $O(\epsilon^2)$ and thus also contributing zero in the limit $\lim_{\epsilon\to 0}\frac{\overset{\vee}{h}(-1+\epsilon)}{\epsilon}$.

Overall we thus get

$$\overset{\vee}{h}'(-1) = \lim_{\epsilon\to 0}\frac{\overset{\vee}{h}(-1+\epsilon)}{\epsilon} = -\sum_{j=1}^{\infty}\tilde{H}_0'(j) \quad . \tag{45}$$

Now by elementary computation, we have that $\frac{\mathrm{d}}{\mathrm{d}z}\left(\cos(\pi z)\right)\big|_{z=j} = 0$ for all j and

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\sin(2\pi z)}{\sin(\frac{\pi z}{n})}\right)\Bigg|_{z=j} = \begin{cases}\frac{2\pi}{\sin(\frac{\pi j}{n})} & , \quad j\notin n\mathbb{Z}_{\geq 0}\\ 0 & , \quad j \text{ a multiple of } n\end{cases}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\sin(\frac{\pi z}{2})\right)\Bigg|_{z=j} = \frac{\pi}{2}\cdot\cos(\frac{\pi j}{2}) = \begin{cases}0 & , \quad j \text{ odd}\\ (-1)^l\cdot\frac{\pi}{2} & , \quad j=2l \text{ even}\end{cases}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{(\frac{z}{n})!}\right)\Bigg|_{z=j} = -\frac{1}{n}\cdot\frac{\psi(\frac{j}{n}+1)}{(\frac{j}{n})!}$$

where $\psi(w) = \frac{\Gamma'(w)}{\Gamma(w)}$ is the well-known di-gamma function, whose values at integer inputs are given by $\psi(0) = -\gamma$ (here $\gamma \approx 0.577\ldots$ is the Euler-Mascheroni constant) and, for $n\in\mathbb{Z}_{>0}$, $\psi(n) = H_n - \gamma$ where $H_n := \sum_{j=1}^{n}\frac{1}{j}$ is the n^{th} harmonic number.

And at the same time we have that $\cos(\pi z)\big|_{z=j} = (-1)^j$ and, understood in the usual way,

$$\frac{\sin(2\pi z)}{\sin(\frac{\pi z}{n})}\Bigg|_{z=j} = \begin{cases}0 & , \quad j\notin n\mathbb{Z}_{\geq 0}\\ (-1)^k\cdot 2n & , \quad j=kn\end{cases}$$

and

$$\sin(\frac{\pi}{2}z)\Big|_{z=j} = \begin{cases}0 & , \quad j \text{ even}\\ (-1)^l & , \quad j=2l+1 \text{ odd}\end{cases}$$

and

$$\frac{1}{(\frac{z}{n})!}\Bigg|_{z=j} = \frac{1}{(\frac{j}{n})!} \quad .$$

Combining these results and applying them when differentiating $\tilde{H}_0(z)$ as defined in equation 43 it follows finally that

$$\begin{aligned} \int_0^\infty \frac{\mathrm{e}^{-x^n}}{1+x^2}\,\mathrm{d}x &= -\overset{\vee}{h}'(-1) \\ &= \left\{ \begin{array}{c} \frac{1}{n}\cdot\sum_{2l+1\notin n\mathbb{Z}}(-1)^{l+1}\cdot\frac{\pi}{\sin\left(\frac{\pi(2l+1)}{n}\right)}\cdot\frac{1}{\left(\frac{2l+1}{n}\right)!} \\ \\ +\frac{1}{n}\cdot\sum_{2l+1=kn}(-1)^k\cdot(-1)^l\frac{\psi(k)}{k!} \\ \\ +\frac{\pi}{2}\cdot\sum_{2l=kn}(-1)^k\cdot(-1)^l\frac{1}{k!} \end{array} \right\} \end{aligned} \qquad (46)$$

where the indices in all these sums are of course restricted to be non-negative.

Comments: (i) This formula for $\int_0^\infty \frac{\mathrm{e}^{-x^n}}{1+x^2}\,\mathrm{d}x$ is rapidly convergent for any $n \in \mathbb{Z}_{>0}$. It is easily confirmed numerically for any given n and is pleasingly elegant for small n. For example, for n even the second sum vanishes, so that for $n = 2$ we get

$$\int_0^\infty \frac{\mathrm{e}^{-x^2}}{1+x^2}\,\mathrm{d}x = \frac{\pi}{2}\cdot\left\{\sum_{l=0}^{\infty}\frac{1}{l!} - \sum_{l=0}^{\infty}\frac{1}{(l+\frac{1}{2})!}\right\}$$

while for $n = 4$ we get

$$\int_0^\infty \frac{\mathrm{e}^{-x^4}}{1+x^2}\,\mathrm{d}x = \frac{\pi}{2}\cdot\left\{\sum_{l=0}^{\infty}\frac{(-1)^l}{l!} - \frac{1}{\sqrt{2}}\cdot\sum_{l=0}^{\infty}\frac{(-1)^l}{(l+\frac{1}{4})!} + \frac{1}{\sqrt{2}}\cdot\sum_{l=0}^{\infty}\frac{(-1)^l}{(l+\frac{3}{4})!}\right\} \quad .$$

There are similar formulae for $n = 3$ and so on.

(ii) This example has been worth detailing not just because it contains a closed-form formula for an integral where it is hard to see how to obtain such a formula by traditional techniques. It is also interesting in showing additional subtleties in the execution of the Taylor-series-to-the-left methodology beyond what we have previously encountered. These subtleties relate both to the method for identifying the necessary TLA-coefficient function as a difference of remainder sums, and to the calculational treatment of those remainder sums where there is s-dependence both in the summands themselves and in their geometric location.

Example Set 3 [$\int_0^\infty \mathrm{e}^{-p(x)}\,\mathrm{d}x$, $p(x)$ any polynomial with positive leading coefficient]: By a simple change of variable we may assume wlog that the leading coefficient of $p(x)$ is 1, and we may also trivially take any constant term in $p(x)$ outside the integral as an overall factor. Thus we shall take the general form of $p(x)$ as

$$p(x) = x^N + c_{N-1}x^{N-1} + \ldots + c_2x^2 + c_1x \quad .$$

In the case where $p(x)$ is the pure power x^N there are already well-known formulae for this integral (see e.g. [1] or [XI]). Likewise, by completing the square, it is easy to obtain a general formula, albeit in terms of the cumulative Gaussian probability function, when $p(x)$ is the quadratic $p(x) = x^2 + c_1 x$.

However, when $p(x)$ has our general form and is of degree greater than 2, general formulae of any character do not appear to be known.[8] We thus now take the case of $deg(p(x)) = 3$ as the first case to consider using our new Taylor-series-to-the-left methods.

Suppose $p(x) = x^3 + bx^2 + ax$. Then if we let $f(x) = \mathrm{e}^{-ax}$, $g(x) = \mathrm{e}^{-bx^2}$ and $h(x) := \mathrm{e}^{-x^3}$ we have that our integrand is given by

$$\mathrm{e}^{-p(x)} = f(x) \cdot g(x) \cdot h(x) \quad . \tag{47}$$

Moreover, we know the TLA-coefficient functions of each of f, g and h, namely:

$$\overset{\vee}{f}(s) = a^s \cdot \frac{\cos(\pi s)}{s!} \quad \text{and} \quad \overset{\vee}{g}(s) = b^{\frac{s}{2}} \cdot \frac{1}{(\frac{s}{2})!} \cdot \cos(\pi s) \cdot \cos(\frac{\pi}{2} s)$$

and

$$\overset{\vee}{h}(s) = \frac{1}{3} \cdot \frac{\sin(\pi s)}{\sin(\frac{\pi s}{3})} \cdot \frac{1}{(\frac{s}{3})!} \cdot \cos(\pi s)$$

and in all three cases the integer-values of these TLA-coefficient functions are entirely attributable to f_0, g_0 and h_0, respectively (i.e. f_∞, g_∞ and h_∞ are all identically zero).

Now, by iteration, lemma 1a in [XII] extends in the natural way from a product of two functions to products of three or more functions. In this case this yields that the TLA-coefficient function of $H(x) := \mathrm{e}^{-p(x)}$ is given by the double-sum

$$\overset{\vee}{H}(s) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \overset{\vee}{f}(j) \cdot \overset{\vee}{g}(k) \cdot \overset{\vee}{h}(s - j - k) \quad . \tag{48}$$

In light of the above, since $\overset{\vee}{f}(j)$ is zero for all $j \in \mathbb{Z}_{<0}$ and $\overset{\vee}{g}(k)$ is zero both for all $k \in \mathbb{Z}_{<0}$ and all $k \in \mathbb{Z}_{\geq 0}$ odd, it follows on writing $k = 2l$ and performing elementary simplifications, that

$$\overset{\vee}{H}(s) = \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \frac{a^j}{j!} \cdot \frac{b^l}{l!} \cdot \frac{(-1)^{j+3l}}{3} \cdot \frac{\sin(\pi s)}{\sin(\frac{\pi}{3}(-j-2l) + \frac{\pi}{3} s)} \cdot \frac{1}{(\frac{(-j-2l)}{3} + \frac{s}{3})!} \cdot \cos(\pi s) \tag{49}$$

Putting $s = -1$ in this equation we find, as we would expect, that $\overset{\vee}{H}(-1) = 0$ since $\frac{\sin(\pi s)}{\sin(\frac{\pi}{3}(-j-2l-1))} = 0$ unless $j + 2l + 1 \equiv 0 \pmod 3$; and when $j + 2l + 1 \equiv 0 \pmod 3$ this expression is finite but $\frac{1}{(\frac{(-j-2l-1)}{3})!} = 0$.

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[8] We are, of course, happy to be corrected on this if in fact we have somehow overlooked them - perhaps they are hiding in a sheltered corner of the Himalayan foothills!

As for $\overset{\vee}{H}(-1+\epsilon)$, we have that

$$\overset{\vee}{H}(-1+\epsilon) = \sum_{j=0}^{\infty}\sum_{l=0}^{\infty} \left\{ \begin{array}{c} \frac{a^j}{j!} \cdot \frac{b^l}{l!} \cdot \frac{(-1)^{j+3l}}{3} \cdot \frac{\sin(-\pi+\pi\epsilon)}{\sin(\frac{\pi}{3}(-j-2l-1)+\frac{\pi\epsilon}{3})} \\ \\ \cdot \frac{1}{(\frac{(-j-2l-1)}{3}+\frac{\epsilon}{3})!} \cdot \cos(-\pi+\pi\epsilon) \end{array} \right\} \quad . \tag{50}$$

Now, working overall to $O(\epsilon^2)$, we have that $\sin(-\pi+\pi\epsilon) = -\pi\epsilon + O(\epsilon^3)$ and

$$\sin(\frac{\pi}{3}(-j-2l-1)+\frac{\pi\epsilon}{3}) = \begin{cases} -\sin\left((j+2l+1)\frac{\pi}{3}\right) + O(\epsilon), & j+2l \neq 2 \pmod 3 \\ (-1)^{r+1}\frac{\pi}{3}\epsilon + O(\epsilon^3), & j+2l = 3r+2 \end{cases}$$

and

$$\frac{1}{(\frac{(-j-2l-1)}{3}+\frac{\epsilon}{3})!} = \begin{cases} \frac{1}{(\frac{(-j-2l-1)}{3})!} + O(\epsilon), & j+2l \neq 2 \pmod 3 \\ (-1)^r \cdot r! \cdot \frac{\epsilon}{3} + O(\epsilon^3), & j+2l = 3r+2 \end{cases}$$

and $\cos(-\pi+\pi\epsilon) = -1 + O(\epsilon^2)$. It follows in equation 50 that for $\overset{\vee}{H}'(-1) = \lim_{\epsilon\to 0} \frac{\overset{\vee}{H}(-1+\epsilon)}{\epsilon}$ we get

$$\overset{\vee}{H}'(-1) = \left\{ \begin{array}{c} \sum\limits_{\substack{j,l=0 \\ j+2l\neq 2 \bmod 3}}^{\infty} \frac{a^j}{j!} \cdot \frac{b^l}{l!} \cdot \frac{(-1)^{j+3l+1}}{3} \cdot \frac{\pi}{\sin(\frac{\pi}{3}(j+2l+1))} \cdot \frac{1}{(\frac{(-j-2l-1)}{3})!} \\ \\ + \sum\limits_{\substack{j,l=0 \\ j+2l=3r+2}}^{\infty} \frac{a^j}{j!} \cdot (-1)^l \cdot \frac{b^l}{l!} \cdot \frac{(-1)^{3r+3}}{3} \cdot r! \end{array} \right\} \quad . \tag{51}$$

This can be expressed more symmetrically by breaking the first double-sum into two cases - for $j+2l=3r$ and $j+2l=3r+1$ respectively. In doing so, we also note that $\sin(r\pi+\frac{\pi}{3}) = (-1)^r\sin(\frac{\pi}{3})$ and $\sin(r\pi+\frac{2\pi}{3}) = (-1)^r\sin(\frac{2\pi}{3})$ and that

$$\frac{1}{(-(r+\frac{1}{3}))!} = \frac{(-1)^r}{(\frac{-1}{3})!} \cdot (\frac{1}{3}) \cdot (\frac{4}{3}) \cdot \cdots \cdot (r-\frac{2}{3})$$

and

$$\frac{1}{(-(r+\frac{2}{3}))!} = \frac{(-1)^r}{(\frac{-2}{3})!} \cdot (\frac{2}{3}) \cdot (\frac{5}{3}) \cdot \cdots \cdot (r-\frac{1}{3}) \quad .$$

Applying these in equation 51 and recalling that $\int_0^\infty \mathrm{e}^{-x^3-bx^2-ax}\,\mathrm{d}x = -\overset{\vee}{H}'(-1)$, we finally obtain the following symmetrised formula for this integral, namely:

$$\int_0^\infty \mathrm{e}^{-x^3-bx^2-ax}\,\mathrm{d}x$$

$$= \quad \frac{1}{3}\cdot\left\{\begin{array}{c} \sum\limits_{\substack{j,l=0\\ j+2l=3r}}^{\infty} (-1)^{l+3r}\cdot\frac{a^j\cdot b^l}{j!\cdot l!}\cdot\frac{\pi}{\sin(\frac{\pi}{3})}\cdot\frac{1}{(\frac{-1}{3})!}\cdot(\frac{1}{3})\cdot(\frac{4}{3})\cdot\dots\cdot(r-\frac{2}{3}) \\ \\ -\sum\limits_{\substack{j,l=0\\ j+2l=3r+1}}^{\infty} (-1)^{l+3r}\cdot\frac{a^j\cdot b^l}{j!\cdot l!}\cdot\frac{\pi}{\sin(\frac{2\pi}{3})}\cdot\frac{1}{(\frac{-2}{3})!}\cdot(\frac{2}{3})\cdot(\frac{5}{3})\cdot\dots\cdot(r-\frac{1}{3}) \\ \\ +\sum\limits_{\substack{j,l=0\\ j+2l=3r+2}}^{\infty} (-1)^{l+3r}\cdot\frac{a^j\cdot b^l}{j!\cdot l!}\cdot r! \end{array}\right\} \tag{52}$$

In identical fashion, for the case of a degree 4 polynomial $p(x) = x^4 + cx^3 + bx^2 + ax$, we derive the formula

$$\int_0^\infty \mathrm{e}^{-x^4-cx^3-bx^2-ax}\,\mathrm{d}x$$

$$= \quad \frac{1}{4}\cdot\left\{\begin{array}{c} \sum\limits_{\substack{j,l,p=0\\ j+2l+3p=4r}}^{\infty} (-1)^{l+4r}\cdot\frac{a^j\cdot b^l\cdot c^p}{j!\cdot l!\cdot p!}\cdot\frac{\pi}{\sin(\frac{\pi}{4})\cdot(\frac{-1}{4})!}\cdot(\frac{1}{4})\cdot\dots\cdot(r-\frac{3}{4}) \\ \\ -\sum\limits_{\substack{j,l,p=0\\ j+2l+3p=4r+1}}^{\infty} (-1)^{l+4r}\cdot\frac{a^j\cdot b^l\cdot c^p}{j!\cdot l!\cdot p!}\cdot\frac{\pi}{\sin(\frac{2\pi}{4})\cdot(\frac{-2}{4})!}\cdot(\frac{2}{4})\cdot\dots\cdot(r-\frac{2}{4}) \\ \\ +\sum\limits_{\substack{j,l,p=0\\ j+2l+3p=4r+2}}^{\infty} (-1)^{l+4r}\cdot\frac{a^j\cdot b^l\cdot c^p}{j!\cdot l!\cdot p!}\cdot\frac{\pi}{\sin(\frac{3\pi}{4})\cdot(\frac{-3}{4})!}\cdot(\frac{3}{4})\cdot\dots\cdot(r-\frac{1}{4}) \\ \\ -\sum\limits_{\substack{j,l,p=0\\ j+2l+3p=4r+3}}^{\infty} (-1)^{l+4r}\cdot\frac{a^j\cdot b^l\cdot c^p}{j!\cdot l!\cdot p!}\cdot r! \end{array}\right\} \tag{53}$$

and for arbitrary $p(x) = x^N + c_{N-1}x^{N-1} + ... + c_2x^2 + c_1x$ of degree N, the corresponding formula for our integral in general is that

$$
\begin{aligned}
&\int_0^\infty \mathrm{e}^{-x^N - c_{N-1}x^{N-1} - \ldots - c_2x^2 - c_1x}\,\mathrm{d}x \\
&= \left\{ \begin{array}{c}
\displaystyle\sum_{q=1}^{N-1} \frac{(-1)^{q-1}}{N} \cdot \sum_{\substack{j_1,\ldots,j_{N-1}=0 \\ j_1+2j_2+\ldots \\ +(N-1)j_{N-1} \\ =Nr+q-1}}^{\infty}
\left\{ \begin{array}{c}
\left[\frac{c_1^{j_1}}{j_1!}\right] \cdot \left[\frac{(-1)^{j_2} c_2^{j_2}}{j_2!}\right] \\
\cdot \left[\frac{c_3^{j_3}}{j_3!}\right] \cdot \left[\frac{(-1)^{j_4} c_4^{j_4}}{j_4!}\right] \cdots \\
\cdot \left[\frac{(-1)^{j_{N-1}\cdot((N-1)(\bmod 2)+1)} c_{N-1}^{j_{N-1}}}{j_{N-1}!}\right] \\
\cdot \frac{\pi}{\sin(\frac{\pi q}{N})\cdot(\frac{-q}{N})!} \cdot (-1)^{Nr} \\
\cdot \left[\left(\frac{q}{N}\right) \cdot \left(\frac{q}{N}+1\right) \cdots \left(\frac{q}{N}+r-1\right)\right]
\end{array} \right\} \\
\displaystyle + \frac{(-1)^{N-1}}{N} \sum_{\substack{j_1,\ldots,j_{N-1}=0 \\ j_1+2j_2+\ldots \\ +(N-1)j_{N-1} \\ =Nr+q-1}}^{\infty}
\left\{ \begin{array}{c}
\left[\frac{c_1^{j_1}}{j_1!}\right] \cdot \left[\frac{(-1)^{j_2} c_2^{j_2}}{j_2!}\right] \\
\cdot \left[\frac{c_3^{j_3}}{j_3!}\right] \cdot \left[\frac{(-1)^{j_4} c_4^{j_4}}{j_4!}\right] \cdots \\
\cdot \left[\frac{(-1)^{j_{N-1}\cdot((N-1)(\bmod 2)+1)} c_{N-1}^{j_{N-1}}}{j_{N-1}!}\right] \\
\cdot (-1)^{Nr} \cdot r!
\end{array} \right\}
\end{array} \right\}
\end{aligned}
\tag{54}
$$

Comments: (i) As noted, the formulae in equations 52 and 53 for degrees 3 and 4 - and the general formula in equation 54 - are, as far as we are aware, new. They give closed-form expressions for $\int_0^\infty \mathrm{e}^{-p(x)}\,\mathrm{d}x$ in terms of sums which are classically convergent, albeit that their convergence is slow and delicate to prove. Despite this slow convergence it is easy, for example, to validate formulae 52 and 53 for suitably small values of a, b and c via numerical integration, and we have done so (e.g. confirming result 52 to 10 d.p.'s for the case of $a = 0.2$ and $b = 0.1$).

(ii) In all cases, and in particular in the general formula 54, the final sum which has been split out could in fact be combined with the earlier $\sum_{q=1}^{N-1}$ to form an expression involving a single overall sum, $\sum_{q=1}^{N}$, as long as we interpret the formally undefined, singular expression $\left(\frac{\pi}{\sin(\frac{\pi q}{N})\cdot(\frac{-q}{N})!}\right)\Big|_{q=N}$ as being $\lim_{\epsilon\to 0}\left(\frac{\pi}{\sin(\frac{\pi q}{N})\cdot(\frac{-q}{N})!}\right)\Big|_{q=N+\epsilon} = 1$ in the spirit we have adopted on many previous occasions in [I]-[XIV]. Doing this makes equation 54 very clean and fully emphasises the symmetrised nature of the formula.

(iii) One application where these formulae might be valuable is in calculating the Mellin transform of a function, $Q(x)$, which is itself given by the exponential of a polynomial in $\ln(x)$ on $[1, \infty)$, i.e.

$$Q(x) = \begin{cases} \mathrm{e}^{-p(\ln x)}, & x \geq 1 \\ 0, & 0 \leq x < 1 \end{cases} .$$

Functions of $\ln x$ for $x \geq 1$ are common in number theory and the theory of ζ, and while they are generally not exponentials of *polynomials* in $\ln x$, approximation by such functions might conceivably be of interest.

For such $Q(x)$, under the change of variables $u = \ln x$ their Mellin transform becomes

$$\mathcal{M}[Q](s) = \int_0^\infty \mathrm{e}^{-p(u)} \cdot \mathrm{e}^{su} \, \mathrm{d}u \tag{55}$$

and so formula 54 gives us an expression for this Mellin transform with the s-dependence captured purely in the adjusted coefficient $\tilde{c}_1(s) = c_1 - s$ and its associated powers $(c_1 - s)^{j_1}$ - effectively giving the Mellin transform as a power series expansion around c_1 in the s-plane.

4 Taylor-series-to-the-left integration across singularities

So far, in discussing Taylor-series-to-the-left methods for integration in [XI]-[XIV], we have considered integrals, $\int_0^\infty f(x) \, \mathrm{d}x$, where $f(x)$ is required to be integrable in either a classical or a generalised Césaro sense on $[0, \infty)$.

As discussed in [I] this includes the possibility of f having Césaro-integrable singularities on $[0, \infty)$, such as poles of order 2 or more. It is only in the case where f has, for example, a pole of order 1 - so that its integral has a Césaro non-convergent log-divergence - that we fall outside this setting[9].

In such cases, however, where f has a pole at $x_0 \in [0, \infty)$ given by $f(x) = \frac{a}{x - x_0} +$ *continuous*, there is a well-established method for still attaching a value to $\int_0^\infty f(x) \, \mathrm{d}x$, namely as the Cauchy p.v. of the integral given by $\lim_{\epsilon \to 0} \left\{ \int_0^{x_0 - \epsilon} f(x) \, \mathrm{d}x + \int_{x_0 + \epsilon}^\infty f(x) \, \mathrm{d}x \right\}$.

As we would hope, it turns out that this Cauchy p.v. definition remains consistent with our Taylor-series-to-the-left methods and results, as long as we use the canonical form of the TLA-coefficient function, $\overset{\vee}{f}(s)$. Moreover, it also turns out that natural alternative approaches to handling such non-Césaro-integrable

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[9] More generally, any function f whose integral would generate divergences which are generalised eigenfunctions of P with eigenvalue 1 would be similarly problematic.

singularities within this framework can be understood as simply corresponding to the use of different gauge-equivalent choices for $\overset{\vee}{f}(s)$.

Let us illustrate this with a class of examples. In [XI] we derived a general formula for $\int_0^\infty \frac{1}{1+x^n}\,\mathrm{d}x$, $n \in \mathbb{Z}_{\geq 2}$, namely

$$\int_0^\infty \frac{1}{1+x^n}\,\mathrm{d}x = \frac{\frac{\pi}{n}}{\sin(\frac{\pi}{n})} \quad . \tag{56}$$

In our Taylor-series-to-the-left methodology this follows trivially from the fact that the canonical TLA-coefficient function of $g(x) := \frac{1}{1+x}$ is $\overset{\vee}{g}(s) = \cos(\pi s)$ and the rule that if $f(x) := g(x^n)$, then the TLA-coefficient functions of f and g are related by $\overset{\vee}{f}(s) = \frac{1}{n} \cdot \frac{\sin(2\pi s)}{\sin(\frac{2\pi s}{n})} \cdot \overset{\vee}{g}(\frac{s}{n})$, which allows us to calculate $\overset{\vee}{f}'(-1)$ easily.

Consider now the case of integrand $f(x) := \frac{1}{1-x^n}$, with a simple pole at $x = 1$ on $[0, \infty)$. Since, for $n = 1$, the function $h(x) := \frac{1}{1-x}$ arises from $g(x) := \frac{1}{1+x}$ by taking $x \mapsto -x$, it is reasonable to take $\overset{\vee}{h}(s) = \overset{\vee}{g}(s) \cdot \cos(\pi s) = \cos^2(\pi s)$ and it then follows that the TLA-coefficient function of $f(x) = \frac{1}{1-x^n}$ is

$$\overset{\vee}{f}(s) = \frac{1}{n} \cdot \frac{\sin(2\pi s)}{\sin(\frac{2\pi s}{n})} \cdot \cos^2(\frac{\pi s}{n}) = \frac{1}{n} \cdot \frac{\sin(\pi s)}{\sin(\frac{\pi s}{n})} \cdot \cos(\pi s) \cdot \cos(\frac{\pi s}{n}) \quad . \tag{57}$$

This has $\overset{\vee}{f}(-1) = 0$ for $n \in \mathbb{Z}_{\geq 2}$ and it follows within our Taylor-series-to-the-left framework that $\int_0^\infty \frac{1}{1-x^n}\,\mathrm{d}x$ should be taken as

$$\int_0^\infty \frac{1}{1-x^n}\,\mathrm{d}x = -\overset{\vee}{f}'(-1) = \left(\frac{\pi}{n}\right) \cdot \cot(\frac{\pi}{n}) \quad . \tag{58}$$

For example, for $n = 2$ we get $\int_0^\infty \frac{1}{1-x^2}\,\mathrm{d}x = 0$ and for $n = 3$ we get $\int_0^\infty \frac{1}{1-x^3}\,\mathrm{d}x = \frac{\pi}{3\sqrt{3}}$, and so on.

But it is easy to check using partial fractions that we likewise have

$$\lim{}_{\epsilon \to 0} \left\{ \int_0^{1-\epsilon} \frac{1}{1-x^2}\,\mathrm{d}x + \int_{1+\epsilon}^\infty \frac{1}{1-x^2}\,\mathrm{d}x \right\} = 0 \tag{59}$$

and

$$\lim{}_{\epsilon \to 0} \left\{ \int_0^{1-\epsilon} \frac{1}{1-x^3}\,\mathrm{d}x + \int_{1+\epsilon}^\infty \frac{1}{1-x^3}\,\mathrm{d}x \right\} = \frac{\pi}{3\sqrt{3}} \tag{60}$$

and so we see that our Taylor-series-to-the-left methodology and integration results continue to hold even where our integrand has simple poles on the integration domain $[0, \infty)$, providing we interpret the associated integral as being given by its Cauchy p.v..

Comments: (i) The fact that these results continue to hold for Cauchy p.v.'s in the presence of singular integrands should not, in fact, be surprising. The reason is effectively hidden in the above derivation.

What makes it work is taking $\overset{\vee}{g}(s) = \cos^2(\pi s)$ as the canonical TLA-coefficient function of $g(x) := \frac{1}{1-x}$ - rather than taking $\overset{\vee}{g}(s)$ as $\cos^4(\pi s)$ or $\cos^6(\pi s)$ or even as the constant function 1, even though these all give equally good replication of the coefficients at $s = m$, $m \in \mathbb{Z}$, arising from the power series expansions for $\frac{1}{1-x}$ around 0 and ∞.

But this choice of $\overset{\vee}{g}(s) = \cos^2(\pi s)$ is equivalent to choosing the Mellin transform of $\frac{1}{1-x}$ as $\mathcal{M}[\frac{1}{1-x}](s) = \frac{\pi}{\tan(\pi s)}$. This is the standard choice found in tables of Mellin transforms and itself reflects the decision to *define* the integral for $\mathcal{M}[\frac{1}{1-x}](s)$, namely $\int_0^\infty \frac{x^{s-1}}{1-x}\,\mathrm{d}x$, as being *given by its Cauchy p.v. across its non-integrable simple pole at* $x = 1$ for arbitrary $s \in \mathbb{C}$.

Thus the use of Cauchy p.v.'s to define integrals across simple poles is embedded at the outset in the standard choice of definition for Mellin transforms and hence, equivalently, of our canonical TLA-coefficient functions. It is therefore natural that Taylor-series-to-the-left integration results based on these TLA-coefficient functions should continue to handle such classically non-integrable simple poles (or other singularities giving rise to Césaro eigenfunction divergences with eigenvalue 1) provided they are defined via Cauchy p.v.'s.

(ii) Of course there are other ways of defining $\int_0^\infty f(x)\,\mathrm{d}x$ around a simple pole of f at x_0. For example, we could do so by defining it as the limit as $\epsilon \to 0$ of the contour integral obtained by replacing the line segment from $x_0 - \epsilon$ to $x_0 + \epsilon$ through the singularity with a half-circle of radius ϵ running clockwise from $x_0 - \epsilon$ to $x_0 + \epsilon$ in the complex plane; or we could take this amended contour as instead running counter-clockwise; and so on.

For example, for $f(x) = \frac{1}{1-x^n}$, since $\frac{1}{1-x^n} = \frac{-1}{(x-1)(x^{n-1}+x^{n-2}+\ldots+1)} \approx \frac{-1}{n} \cdot \frac{1}{(x-1)}$ +*continuous* for x near 1, it is easy to see that in the first case the amended contour would add an imaginary component of $\frac{i\pi}{n}$ to our integral, changing our value under this definition to

$$\int_0^\infty f(x)\,\mathrm{d}x = \frac{\pi}{n} \cdot \cot(\frac{\pi}{n}) + \frac{i\pi}{n} \tag{61}$$

while in the second case it would subtract this imaginary component and leave

$$\int_0^\infty f(x)\,\mathrm{d}x = \frac{\pi}{n} \cdot \cot(\frac{\pi}{n}) - \frac{i\pi}{n} \quad . \tag{62}$$

How do these alternatives play out in our framework of Taylor-series-to-the-left methodology and TLA-coefficient functions? Well, as discussed in [XI], a TLA-coefficient function, $\overset{\vee}{f}(s)$, is only initially constrained to have prescribed values at integer points, so there is considerable gauge-freedom in how to assign $\overset{\vee}{f}(s)$ to f.

If for $f(x) = \frac{1}{1-x^n}$ we use the canonical choice given by $\overset{\vee}{f}(s) = -\mathcal{M}[f](-s) \cdot \frac{\sin(2\pi s)}{2\pi} = \frac{1}{n} \cdot \frac{\sin(\pi s)}{\sin(\frac{\pi s}{n})} \cdot \cos(\pi s) \cdot \cos(\frac{\pi s}{n})$ then we end up with $\int_0^\infty f(x)\, \mathrm{d}x = \frac{\pi}{n} \cdot \cot(\frac{\pi}{n})$ as derived above.

But taking $\overset{\vee}{f}(s)$ instead as

$$\overset{\vee}{f}(s) = \frac{1}{n} \cdot \frac{\sin(\pi s)}{\sin(\frac{\pi s}{n})} \cdot \cos(\pi s) \cdot \mathrm{e}^{-\frac{i\pi s}{n}}$$

or

$$\overset{\vee}{f}(s) = \frac{1}{n} \cdot \frac{\sin(\pi s)}{\sin(\frac{\pi s}{n})} \cdot \cos(\pi s) \cdot \mathrm{e}^{\frac{i\pi s}{n}}$$

are perfectly good alternative gauge-equivalent choices (their values agree appropriately with the coefficients in the power series expansions of $f(x) = \frac{1}{1-x^n}$ around 0 and around ∞) and it is trivial to see that under the first of these we obtain instead that

$$\int_0^\infty f(x)\, \mathrm{d}x = \frac{\pi}{n} \cdot \csc(\frac{\pi}{n}) \cdot \mathrm{e}^{\frac{i\pi}{n}} = \frac{\pi}{n} \cdot \cot(\frac{\pi}{n}) + \frac{i\pi}{n} \quad ,$$

while the second would give

$$\int_0^\infty f(x)\, \mathrm{d}x = \frac{\pi}{n} \cdot \csc(\frac{\pi}{n}) \cdot \mathrm{e}^{\frac{-i\pi}{n}} = \frac{\pi}{n} \cdot \cot(\frac{\pi}{n}) - \frac{i\pi}{n} \quad .$$

Thus our alternative contour-based approaches to defining $\int_0^\infty f(x)\, \mathrm{d}x$ across its pole at $x = 1$ correspond within Taylor-series-to-the-left methodology simply to making different gauge-equivalent choices of TLA-coefficient function, $\overset{\vee}{f}(s)$.

Overall, we see that the gauge-freedom inherent in the definition of TLA-coefficient functions in fact matches the intrinsic ambiguity which arises within the generalised Césaro context in dealing with log-divergences in integration. Yet at the same time there is a *canonical* choice of TLA-coefficient function which embeds the choice of using Cauchy p.v.'s at its heart (via the definition of Mellin transforms for functions with simple poles on $[0, \infty)$) and which can be used as the designated way to handle this ambiguity.

5 Towards a calculus of TLA-coefficient functions

It is natural to ask whether we can identify a full "calculus" for TLA-coefficient functions, identifying rules for how they behave under natural operations on underlying functions - such as taking products, quotients, composition, exponentiation, differentiation and so forth.

5.1 A hybrid calculus

Rules of this kind are already known for Mellin transforms and these could, of course, simply be translated across mechanically and re-phrased in terms of TLA-coefficient functions using the fundamental relationship between the canonical TLA-coefficient of a function and its Mellin transform.

But, as discussed in [XII, section 2.4], these rules for Mellin transforms are often somewhat abstract and difficult to work with. Direct use of TLA-coefficient functions may instead offer greater simplicity and practical utility.

For example, already in this paper (and on many earlier occasions in [XI]-[XIV]) we have used what is effectively a product rule for TLA-coefficient functions, namely that if $h(x) = f(x) \cdot g(x)$ then

$$\overset{\vee}{h}(s) = \sum_{j=-\infty}^{\infty} f_0(j) \cdot g_0(s-j) - \sum_{j=-\infty}^{\infty} f_\infty(j) \cdot g_\infty(s-j) \tag{63}$$

in order to carry out computations which seem otherwise intractable. Indeed, in example set 3 in section 3, we used a straightforward extension of this rule to products of three or more underlying functions.

At the same time, in doing so there has been a certain amount of practical "muddling through" which is hard to refine into clear, simple rules - for example, in how to handle the transition from $s = m \in \mathbb{Z}$ to arbitrary $s \in \mathbb{C}$ in the calculations for example set 2 in section 3. These first necessitated that we rewrite the finite sum which arose when $s = m \in \mathbb{Z}$ into a difference of two remainder sums before moving to arbitrary s; and then in turn required care in handling the presence of s-dependence in both the function to which the operators $R_{+,0}$ and R_+ were applied, and in the geometric location at which these remainder sums were to be taken.

Nonetheless, this muddling through merely entailed applying principles which are natural in the generalised geometric Césaro context; and overall it could be performed much more directly and easily than by trying instead to work via the corresponding product rule for Mellin transforms - which involves a contour integral convolution, namely that

$$\mathcal{M}[f(x) \cdot g(x)](s) = \frac{1}{2\pi i} \cdot \int_{c-i\infty}^{c+i\infty} \tilde{f}(r) \cdot \tilde{g}(s-r)\,\mathrm{d}r \tag{64}$$

where $\tilde{f}(s) := \mathcal{M}[f](s)$ and $\tilde{g}(s) := \mathcal{M}[g](s)$ are the Mellin transforms of f and g respectively.[10]

Overall, our general view is thus that it *is* often easier to work with TLA-coefficient functions and Taylor-series-to-the-left methodology within a generalised Césaro context than to work via existing Mellin transform laws; but that this is perhaps best done not by seeking a complete, all-encompassing "calculus" of such TLA-coefficient functions, but rather by adopting a hybrid perspective

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[10] Note that extension of such contour integral convolution to products of three or more functions as required in example set 3 would pose further challenges in execution.

which includes some clear-cut laws mixed together with a range of principles, techniques and tricks that are practical and effective, albeit harder to encode as stand-alone results.

For example, among the clear-cut rules we have in this hybrid approach are the product rule for canonical TLA-coefficient functions just discussed; also the rule used in section 4 (namely that if $f(x) := g(x^n)$ then $\overset{\vee}{f}(s) = \frac{1}{n} \cdot \frac{\sin(2\pi s)}{\sin(\frac{2\pi s}{n})} \cdot \overset{\vee}{g}(\frac{s}{n})$; many other simple rules such as that if $f(x) := x^n \cdot g(x)$ then $\overset{\vee}{f}(s) = \overset{\vee}{g}(s-n)$, and that if $f(x) := g'(x)$ then $\overset{\vee}{f}(s) = (s+1) \cdot \overset{\vee}{g}(s+1)$; and so on.

But applying these rules will often, as in section 3, entail also invoking techniques such as how to generalise finite sums within the geometric Césaro context by interpreting $\sum_{j=0}^{s} f(j)$ as $R_{+,0}[f](0) - R_{+}[f](s)$ when s ceases to be an integer and becomes an arbitrary complex parameter.

Indeed, one of the most helpful of these general hybrid principles will be simply to remember the relationship between a canonical TLA-coefficient function, $\overset{\vee}{f}(s)$, at its integer-values and the functions f_0 and f_∞ encoding the coefficients in the power series for f around $x = 0$ and as $x \to \infty$ - namely that if

$$f(x) = \sum_{j=-\infty}^{\infty} f_0(j) x^j + \mathcal{S}_0(x) \quad \text{and} \quad f(x) = \sum_{j=-\infty}^{\infty} f_\infty(j) x^j + \mathcal{S}_\infty(x)$$

for x near 0 and as $x \to \infty$ respectively, then for all $s = m \in \mathbb{Z}$ we have that

$$\overset{\vee}{f}(m) = f_0(m) - f_\infty(m) \quad .$$

This is because one of the principal reasons why working with TLA-coefficient functions can often be preferable to working via Mellin transforms is precisely because it is easier to get hold of these expansions for f near 0 and ∞ than it is to get hold of the general form of the Mellin transform of f. This thus allows us to identify f_0 and f_∞ directly and hence deduce $\overset{\vee}{f}$ at its integer-values $s = m \in \mathbb{Z}$, after which we may then be able to pass to arbitrary $s \in \mathbb{C}$ using the remainder-summation trick just described or some similar technique. By contrast, there are no obvious corresponding results allowing us to get a foothold - and progress to a final answer via such smaller, intermediate steps - when working in the world of Mellin transforms.

In the same way, when considering the TLA-coefficient function of a *quotient* of underlying functions, i.e. $h(x) := \frac{f(x)}{g(x)}$, rather than seek a general "quotient rule" to mirror the product rule already mentioned, we might instead work within our hybrid approach by using the principle just mentioned to get to a point where we can either calculate directly or else invoke the TLA-product rule. That is, we might first seek to break f and g down to deduce their power series components - f_0 and f_∞ on the one hand; g_0 and g_∞ on the other. Next, writing $\tilde{g}(x) := \frac{1}{g(x)}$, we might try to use standard methods to deduce $\tilde{g}_0$ and $\tilde{g}_\infty$ in terms of g_0 and g_∞ respectively. And then finally we might use our product rule for $h(x) := f(x)\cdot\tilde{g}(x)$ to obtain a general form for $\overset{\vee}{h}(s)$ from this knowledge of f_0, $\tilde{g}_0$, f_∞ and $\tilde{g}_\infty$, perhaps using our remainder-summation trick or some similar technique, as required.

Of course none of this is *guaranteed* to work, and certainly not to be straightforward or trivial to carry out. But this hybrid TLA-calculus approach does at least offer a path forward in cases where we cannot even get started on the Mellin transform side; and what may be an easier such path even in cases where progress using Mellin transforms directly is possible.

5.2 One further tool in the hybrid approach - formal symbols and formal function elements

We conclude our discussion of the potential value and advantages of this proposed hybrid calculus for TLA-coefficient functions with one final observation. It is that a further possible advantage may lie in being able to use formal symbols and formal function elements - as introduced in [III] - within this framework.

We illustrate this with an example, one which also demonstrates some of the elements of our hybrid approach just discussed.

Recall that in [XI] we considered the function $f(x) = \mathrm{e}^{-\frac{1}{1+x^2}}$ and used its TLA-coefficient function and Taylor-series-to-the-left methods to deduce a rapidly convergent closed-form formula for the Césaro-value of $\int_0^\infty \mathrm{e}^{-\frac{1}{1+x^2}}\,\mathrm{d}x$ (which is the same as the classical value of $\int_0^\infty (\mathrm{e}^{-\frac{1}{1+x^2}} - 1)\,\mathrm{d}x$) as

$$\int_0^\infty \mathrm{e}^{-\frac{1}{1+x^2}}\,\mathrm{d}x = -\frac{\pi}{2}\cdot\left\{\begin{array}{c} 1 - \frac{1}{2!}\left(\frac{1}{2}\right) + \frac{1}{3!}\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\frac{1}{2!} \\ -\frac{1}{4!}\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\frac{1}{3!} + \ldots \end{array}\right\} \approx -1.258924257 \quad .$$

This integral evaluation is not straightforward by traditional means and we worked by (a) expanding $\mathrm{e}^{-\frac{1}{1+x^2}}$ as $1 - \frac{1}{1+x^2} + \frac{1}{2!}\frac{1}{(1+x^2)^2} - \frac{1}{3!}\frac{1}{(1+x^2)^3} + \ldots$; then (b) expanding each term $(-1)^n\frac{1}{(1+x^2)^n}$ in its Taylor series around 0; (c) combining terms of the same degree to find a formula for $\overset{\vee}{f}(m)$, $m \in \mathbb{Z}_{\geq 0}$, and hence for $\overset{\vee}{f}(s)$, $s \in \mathbb{C}$ arbitrary; and finally (d) evaluating $\int_0^\infty \mathrm{e}^{-\frac{1}{1+x^2}}\,\mathrm{d}x$ as $-\overset{\vee}{f}{}'(-1)$.

Suppose in step (b) we instead work by focussing on $m \in \mathbb{Z}_{<0}$ and expanding around $x = \infty$ rather than $x = 0$. Then an interesting thing occurs.

Since the expansion $\frac{1}{(1+x^2)^n} = \frac{1}{x^{2n}}\cdot\frac{1}{(1+\frac{1}{x^2})^n} = x^{-2n} - nx^{-2n-2} + \ldots$ starts at degree $-2n$, so for any $m \in \mathbb{Z}_{<0}$ there are only finitely many terms which contribute to the coefficient of x^{-m}. In fact for $m = -2l$ it is easy to see that $\overset{\vee}{f}(-2l) = -f_\infty(-2l)$ is given by

$$\overset{\vee}{f}(-2l) = (-1)^{l+1}\cdot\left\{1\cdot\frac{1}{1!} + \binom{l-1}{1}\cdot\frac{1}{2!} + \binom{l-1}{2}\cdot\frac{1}{3!} + \ldots + \binom{l-1}{l-1}\cdot\frac{1}{l!}\right\} .$$

Letting f be the formal function element for the factorial function, defined by

$$f^s := s! = \Gamma(s+1) \tag{65}$$

this can be expressed neatly as

$$\overset{\vee}{f}(-2l) = (-1)^{l+1} \cdot \frac{1}{f} \cdot \left(1 + \frac{1}{f}\right)^{l-1}$$

and, in light of our lemmas from [XI], this suggests that in general we have

$$\overset{\vee}{f}(s) = -\cos(\pi s) \cdot \cos(\frac{\pi}{2} s) \cdot \frac{1}{f} \cdot \left(1 + \frac{1}{f}\right)^{\frac{-s}{2}-1} \quad .$$

But it then follows immediately that $\overset{\vee}{f}(-1) = 0$ and that

$$\begin{aligned} \overset{\vee}{f}'(-1) \quad &= \quad \frac{\pi}{2} \cdot \frac{1}{f} \cdot \left(1 + \frac{1}{f}\right)^{-\frac{1}{2}} = \frac{\pi}{2} \cdot \left\{ \frac{1}{f} + \binom{\frac{-1}{2}}{1} \cdot \frac{1}{f^2} + \binom{\frac{-1}{2}}{2} \cdot \frac{1}{f^3} + \ldots \right\} \\ &= \quad \frac{\pi}{2} \cdot \left\{ \begin{array}{l} 1 - \frac{1}{2!}\left(\frac{1}{2}\right) + \frac{1}{3!}\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\frac{1}{2!} \\ \\ -\frac{1}{4!}\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\frac{1}{3!} + \ldots \end{array} \right\} \end{aligned}$$

and this at once gives us $\int_0^\infty \mathrm{e}^{-\frac{1}{1+x^2}}\, \mathrm{d}x = -\overset{\vee}{f}'(-1) \approx -1.258924257$ in agreement with our earlier calculation.[11]

Comment: We see that here the use of a formal function element has simplified both the derivation and the expression of the TLA-coefficient function we are studying, and in so doing has allowed a simpler calculation of the (Césaro value of the) integral $\int_0^\infty \mathrm{e}^{-\frac{1}{1+x^2}}\, \mathrm{d}x$ under our Taylor-series-to-the-left methodology.

In fact, the argument just given works identically for $\int_0^\infty \mathrm{e}^{-\frac{1}{1+x^k}}\, \mathrm{d}x$ for arbitrary $k \in \mathbb{Z}_{\geq 1}$ to give

$$\int_0^\infty \left(exp\left(\frac{-1}{1+x^k}\right)\right) dx = -\frac{\frac{\pi}{k}}{\sin\left(\frac{\pi}{k}\right)} \cdot \left\{ \begin{array}{c} \frac{1}{1!}\binom{\frac{1}{k}-1}{0} + \frac{1}{2!}\binom{\frac{1}{k}-1}{1} \\ \\ +\frac{1}{3!}\binom{\frac{1}{k}-1}{2} + \ldots \end{array} \right\} \tag{66}$$

in just a few lines of elementary algebra and combinatorics, despite the fact that (to the author at least) it is extremely unclear how to obtain this formula for arbitrary k by traditional integration methods.

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[11] Note here that, as always, we need to be careful in working with formal function elements. We need to ensure that we only evaluate them after combining them fully in each term (i.e. we need to take $\frac{1}{f^j} \cdot \frac{1}{f^k}$ as $\frac{1}{f^{j+k}} = \frac{1}{(j+k)!}$, not as $\frac{1}{j!} \cdot \frac{1}{k!}$) and we need to remember to apply exponents directly (for example, $\frac{1}{f^{-3}}$ means $\frac{1}{(-3)!} = 0$, not $f^3 = 3!$); and so on.

As such, we believe more generally that invocation of formal symbols and formal function elements may be useful in many cases when working with TLA-coefficient functions; and may represent a further advantage of working with TLA-coefficient functions vis-a-vis working with Mellin transforms (where it is much harder to envisage the use of such formal quantities as aids to calculation).

6 Remaining challenges

We have seen that Taylor-series-to-the-left methodology and a hybrid calculus of TLA-coefficient functions facilitates calculation of many results, including integrals, which are challenging using traditional means. It is not hard, however, to find cases which remain challenging and where these methods run up against new limitations. We consider some examples now.

In [XI] we took the case of $f(z) := \frac{1}{1+z^2}$ and calculated that $\int_0^\infty f(z)\mathrm{d}z = \frac{\pi}{2}$ by deducing that $\overset{\vee}{f}(s) = \cos(\pi s)\cos(\frac{\pi}{2}s)$ and evaluating the integral as $-\overset{\vee}{f}'(-1)$.

Suppose we next consider

$$g(z) := \frac{1}{1+z^2\ln z} \quad . \tag{67}$$

This minor amendment causes great difficulty for Taylor-series-to-the-left methodology because it introduces into the power series for g near 0 and ∞ powers of $z^n(\ln z)^k$ where $k \in \mathbb{Z}$ is arbitrarily large, both positive and negative. This makes the determination of the canonical TLA-coefficient function, $\overset{\vee}{g}(s)$, problematic - so much so that we have so far been unable to find a simple, usable expression for $\overset{\vee}{g}(s)$.

To explore this in more detail, note that the positive powers occur in the power series around 0 where we have $g(z) \approx 1 - z^2\ln z + z^4(\ln z)^2 - \ldots$; while the negative powers occur in the power series as $z \to \infty$ where we have $g(z) \approx \frac{1}{z^2\ln z} - \frac{1}{z^4(\ln z)^2} + \ldots$.

Taking the power series around 0 first, the constant term means we should have $\overset{\vee}{g}(0) = 1$; and for each $m = 2l \in \mathbb{Z}_{>0}$ the presence of the term $(-1)^l z^{2l}(\ln z)^l$ means that for s near $2l$ we should have

$$\overset{\vee}{g}(s) \approx (-1)^l \cdot \frac{l!}{(s-2l)^l} + \mathit{continuous} \quad . \tag{68}$$

This is because if the power series for $g(z)$ simply had a term $(-1)^l z^{2l}$, this would mean that $\overset{\vee}{g}(2l) = (-1)^l$ and thus that $-\mathcal{M}[g](-s) = (-1)^l \cdot \frac{2\pi}{\sin(2\pi s)} \approx (-1)^l \cdot \frac{1}{(s-2l)} + \mathit{continuous}$. The additional factor of $(\ln z)^l$ in the actual term in the power series for $g(z)$ corresponds to $(-1)^l$ times l-fold differentiation in the Mellin transform, $-\mathcal{M}[g](-s)$, and we thus actually get $-\mathcal{M}[g](-s) \approx (-1)^l \cdot \frac{l!}{(s-2l)^{l+1}} + \mathit{continuous}$. Multiplication by $\frac{\sin(2\pi s)}{2\pi}$ then finally gets us to equation 68, on passing back from the Mellin transform to the canonical TLA-coefficient function $\overset{\vee}{g}(s)$.

We see immediately from equation 68 that the form of $\overset{\vee}{g}(s)$ is more challenging to deduce than we have previously encountered, because it has singularities of arbitrarily high order, and the order of these singularities is also a function of their location (the singularity of $\overset{\vee}{g}(s)$ near $s = 2l$ is of order l).

If we let M be the operator of multiplication by $\frac{\sin(2\pi s)}{2\pi}$ on functions on s-space, and we then let $A := M \circ \left(-\frac{\mathrm{d}}{\mathrm{d}s}\right) \circ M^{-1}$ be the conjugate operator just used, we might speculate that $\overset{\vee}{g}(s)$ should be related to the TLA-coefficient function $\overset{\vee}{f}(s) = \cos(\pi s) \cdot \cos(\frac{\pi}{2}s)$ of $f(z)$, by

$$\overset{\vee}{g}(s) = A^s[\overset{\vee}{f}](s) = \frac{\sin(2\pi s)}{2\pi} \cdot \left(-\frac{\mathrm{d}}{\mathrm{d}s}\right)^s \left[\frac{\frac{\pi}{2}}{\sin(\frac{\pi}{2}s)}\right] \tag{69}$$

but this would lead us into the realm of dealing with non-integral, indeed arbitrary complex, powers of A and therefore of $\frac{\mathrm{d}}{\mathrm{d}s}$. This is something we have done before in previous papers (in dealing with root identities and in other contexts) and we believe is well worth pursuing again here. We omit any such further exploration for now, however, other than to reiterate that we have not so far been able thereby to derive a clean, usable form for $\overset{\vee}{g}(s)$.

Instead, let us work from a consideration of $\int_0^\infty g(z)\mathrm{d}z$ and use a change of variables to refine these difficulties regarding $\overset{\vee}{g}(s)$ into a different but related set of challenges for Taylor-series-to-the-left methodology - ones which are less abstract and more accessible, simple and direct.

Writing $z = \mathrm{e}^u$ and then $v = -u$ we have that

$$\int_0^\infty \frac{1}{1+z^2\ln z}\mathrm{d}z = \int_{-\infty}^\infty \frac{\mathrm{e}^u}{1+u\mathrm{e}^{2u}}\mathrm{d}u = \int_0^\infty \frac{\mathrm{e}^u}{1+u\mathrm{e}^{2u}}\mathrm{d}u + \int_0^\infty \frac{\mathrm{e}^{-v}}{1-v\mathrm{e}^{-2v}}\mathrm{d}v$$

and both these integrals are now over $[0,\infty)$ and therefore immediately set up for application of Taylor-series-to-the-left methods.

Letting

$$h_1(u) := \frac{\mathrm{e}^u}{1+u\mathrm{e}^{2u}} \quad \text{and} \quad h_2(v) := \frac{\mathrm{e}^{-v}}{1-v\mathrm{e}^{-2v}} \tag{70}$$

we see that both functions are Schwartzian near ∞ and have well-defined Taylor series in neighbourhoods of 0. Moreover, we have $h_2(v) = h_1(-v)$ so that these two Taylor series near 0 have coefficients which agree for even powers and are negatives of each other for odd powers - i.e. $\overset{\vee}{h}_2(m) = \overset{\vee}{h}_1(m)$ for $m \in \mathbb{Z}_{\geq 0}$ even, and $\overset{\vee}{h}_2(m) = -\overset{\vee}{h}_1(m)$ for $m \in \mathbb{Z}_{\geq 0}$ odd.

Now for u small we have that $\frac{\mathrm{e}^u}{1+u\mathrm{e}^{2u}} = \mathrm{e}^u - u\mathrm{e}^{3u} + u^2\mathrm{e}^{5u} - u^3\mathrm{e}^{7u} + \ldots$ and it is easy to derive that the Taylor series for $h_1(u)$ is

$$h_1(u) = \left\{ \begin{array}{c} 1 + \left[\frac{1}{1!}1^1 - \frac{1}{0!}3^0\right] \cdot u + \left[\frac{1}{2!}1^2 - \frac{1}{1!}3^1 + \frac{1}{0!}5^0\right] \cdot u^2 \\ + \left[\frac{1}{3!}1^3 - \frac{1}{2!}3^2 + \frac{1}{1!}5^1 - \frac{1}{0!}7^0\right] \cdot u^3 \\ + \left[\frac{1}{4!}1^4 - \frac{1}{3!}3^3 + \frac{1}{2!}5^2 - \frac{1}{1!}7^1 + \frac{1}{0!}9^0\right] \cdot u^4 + \dots \end{array} \right\} \quad . \tag{71}$$

Noting that h_1 is Schwartzian as $u \to \infty$, we see that in general, for $m \in \mathbb{Z}_{\geq 0}$ we have

$$\overset{\vee}{h}_1(m) = \left\{ \begin{array}{c} \frac{1}{m!}1^m - \frac{1}{(m-1)!}3^{m-1} + \frac{1}{(m-2)!}5^{m-2} - \dots \\ +(-1)^j \frac{1}{(m-j)!}(2j+1)^{m-j} + \dots + (-1)^m \frac{1}{0!}(2m+1)^0 \end{array} \right\} \tag{72}$$

and then, as discussed,

$$\overset{\vee}{h}_2(m) = \cos(\pi m) \cdot \overset{\vee}{h}_1(m) \qquad , \quad m \in \mathbb{Z}_{\geq 0} \quad . \tag{73}$$

Unfortunately, it is not straightforward to move from $m \in \mathbb{Z}_{\geq 0}$ to arbitrary $s \in \mathbb{C}$ in the manner we have done before in equation 72, because the finite sum involved cannot be readily viewed as a difference of remainder sums on an underlying fabric.

Nonetheless, we could instead view formula 72 as extending indefinitely - with the additional terms $(-1)^{m+1}\frac{1}{(-1)!}(2m+3)^{-1} + (-1)^{m+2}\frac{1}{(-2)!}(2m+5)^{-2} + \dots$ all being identically zero - and hence try to generalise to arbitrary $s \in \mathbb{C}$ as

$$\overset{\vee}{h}_1(s) = \frac{1}{s!}1^s - \frac{1}{(s-1)!}3^{s-1} + \frac{1}{(s-2)!}5^{s-2} - \dots \tag{74}$$

with

$$\overset{\vee}{h}_2(s) = \cos(\pi s) \cdot \overset{\vee}{h}_1(s) \quad . \tag{75}$$

Then it is immediate that $\overset{\vee}{h}_1(-1) = 0$ and since $\frac{1}{(-1+\epsilon)!} \approx \epsilon + O(\epsilon^2)$ and $\frac{1}{(-j+\epsilon)!} \approx (-1)^{j-1}(j-1)!\,\epsilon + O(\epsilon^2)$, it follows that

$$\overset{\vee}{h}{}_1'(-1) = \lim_{\epsilon \to 0} \frac{\overset{\vee}{h}_1(-1+\epsilon)}{\epsilon} = 1 + \frac{1}{3^2} + \frac{2!}{5^3} + \frac{3!}{7^4} + \frac{4!}{9^5} + \dots \approx 1.130098101522 \quad . \tag{76}$$

But in fact, by numerical integration we have that

$$\int_0^\infty \frac{\mathrm{e}^u}{1 + u\mathrm{e}^{2u}} \mathrm{d}u \approx 0.901205760451 \tag{77}$$

and so this guess at the form of $\overset{\vee}{h}_1(s)$ must be *wrong*, since we do not have $\int_0^\infty h_1(u)\mathrm{d}u = -\overset{\vee}{h}{}_1'(-1)$, either in magnitude or sign.

Interestingly, however, this guess at $\overset{\vee}{h}_1(s)$ together with equation 75 would give us $\overset{\vee}{h}_2(-1) = 0$ and $\overset{\vee}{h}'_2(-1) = -\left\{1 + \frac{1}{3^2} + \frac{2!}{5^3} + \ldots\right\} \approx -1.130098101522$, and in this case the conclusion from our Taylor-series-to-the-left methodology that

$$\int_0^\infty \frac{\mathrm{e}^{-v}}{1 - v\mathrm{e}^{2v}}\mathrm{d}v = -\overset{\vee}{h}'_2(-1) \approx 1.130098101522 \tag{78}$$

turns out to be *correct*, confirmed to 12 d.p.'s by numerical integration.

Thus it appears that our guess at the form of $\overset{\vee}{h}_2(s)$ here as

$$\overset{\vee}{h}_2(s) = \cos(\pi s) \cdot \left\{\frac{1}{s!}1^s - \frac{1}{(s-1)!}3^{s-1} + \frac{1}{(s-2)!}5^{s-2} - \ldots\right\} \tag{79}$$

is correct; but that the corresponding guess for $\overset{\vee}{h}_1(s)$ in equation 74, based on the (up to sign) identical formulae for $\overset{\vee}{h}_1(m)$ and $\overset{\vee}{h}_2(m)$ when $s = m \in \mathbb{Z}_{\geq 0}$, is not correct.

Now, note that this error regarding $h_1(u)$ and its integral is not a failure of Taylor-series-to-the-left methodology per se. The Mellin transform

$$\begin{aligned} -\mathcal{M}[h_1](-(-1+\epsilon)) &= -\int_0^\infty \frac{\mathrm{e}^u \cdot u^{-\epsilon}}{1 + u\mathrm{e}^{2u}}\mathrm{d}u \\ &= -0.901205760451 + \epsilon \cdot \int_0^\infty \frac{\mathrm{e}^u \cdot \ln u}{1 + u\mathrm{e}^{2u}}\mathrm{d}u + O(\epsilon^2) \end{aligned}$$

is finite and smooth for ϵ small so that, whatever $-\mathcal{M}[h_1](-s)$ is as a function, the canonical TLA-coefficient function of h_1 given by

$$\overset{\vee}{h}_1(s) = -\mathcal{M}[h_1](-s) \cdot \frac{\sin(2\pi s)}{2\pi}$$

does satisfy that

$$\int_0^\infty h_1(u)\mathrm{d}u = \int_0^\infty \frac{\mathrm{e}^u}{1 + u\mathrm{e}^{2u}}\mathrm{d}u = -\overset{\vee}{h}'_1(-1) \quad . \tag{80}$$

Rather, the proper conclusion to draw is that we do not know how to calculate $-\mathcal{M}[h_1](-s)$ directly; and that our intuitive attempt to instead infer $\overset{\vee}{h}_1(s)$ for arbitrary $s \in \mathbb{C}$ in equation 74, based on the form of $\overset{\vee}{h}_1(m)$ when $m \in \mathbb{Z}_{\geq 0}$ in equation 72, fails to give the correct canonical form of $\overset{\vee}{h}_1(s)$ - even though the exact same reasoning does give the correct form for $\overset{\vee}{h}_2(s)$.

In other words, the right conclusion to draw is that, while our hybrid calculus of TLA-coefficient functions can be powerful, it remains something of an art - and certainly far from being a straightforwardly mechanical process. There remain many cases where it is difficult to implement and where even seemingly natural methods for inferring a TLA-coefficient function at general $s \in \mathbb{C}$ from its known form for $s = m \in \mathbb{Z}_{\geq 0}$ can go awry[12].

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[12] or, to use the correct technical term "gang aft agley"

On the positive side, however, this leaves open many questions regarding how to make such inference more reliable. Towards this end, we conclude this section with a few additional comments regarding the examples we have considered.

Comments: **(i)** In the last example, why did our method of inference work for $h_2(u)$ but not for $h_1(u)$?

One difference between $h_1(u)$ and $h_2(u)$ which is possibly relevant, is that $v\mathrm{e}^{-2v}$ is bounded less than $\frac{1}{2\mathrm{e}} < 1$ on $[0, \infty)$, so that expanding $\frac{\mathrm{e}^{-v}}{1-v\mathrm{e}^{-2v}}$ as $\mathrm{e}^{-v}+v\mathrm{e}^{-3v}+v^2\mathrm{e}^{-5v}+\ldots$ applies on all of $[0, \infty)$. By contrast, $u\mathrm{e}^{2u}$ is unbounded on $[0, \infty)$ and in a way which involves exponential rather than merely power divergences, so that the expansion we used earlier of $\frac{\mathrm{e}^{u}}{1+u\mathrm{e}^{2u}}$ as $\mathrm{e}^{u} - u\mathrm{e}^{3u} + u^2\mathrm{e}^{5u} + \ldots$ is only local around 0.

Of course, this does not affect the correctness of the Taylor series we derived for $h_1(u)$ and $h_2(u)$ around 0 in equations 71-73; and in the past our Taylor-series-to-the-left integration results on $[0, \infty)$ have applied even where such Taylor series had only finite radii of convergence.

But perhaps here the fact that the resulting expansion for $h_2(u)$ involves a sum of terms of the form $v^j\mathrm{e}^{-(2j+1)v}$, and that all of these themselves satisfy our fundamental Taylor-series-to-the-left integration rule relating their integral on all of $[0, \infty)$ to the negative of the derivative of their TLA-coefficient function at $s = -1$, is relevant in leading to $h_2(v)$ itself also satisfying this relationship with the natural choice of $\overset{\vee}{h}_2(s)$ as its canonical TLA-coefficient function.

By contrast, the corresponding sum in the case of $h_1(u)$ involves terms of the form $u^j\mathrm{e}^{+(2j+1)u}$, none of which are convergent on all of $[0, \infty)$ even in a generalised Césaro sense and therefore none of which satisfy this fundamental Taylor-series-to-the-left integration rule on $[0, \infty)$.

On this point, it is interesting to note in passing that functions of the form e^{+ku} or $u^l\mathrm{e}^{+ku}$ can of course still be assigned canonical TLA-coefficient functions (namely $\frac{k^s}{s!} \cdot \cos^2(\pi s)$ and $\frac{k^{s-l}}{(s-l)!} \cdot \cos^2(\pi(s - l))$ respectively); and that when we do this we can in fact state a variant of our fundamental Taylor-series-to-the-left integration result which does apply for these functions.

Specifically, in this case, the negative of the derivative of these TLA-coefficient functions evaluated at $s = -1$ gives us the classically-convergent value of the integral of these functions from 0 to $-\infty$ (i.e. as $\int_0^{-\infty}$).

Interpreted this way, the intuitive form we guessed for $\overset{\vee}{h}_1(s)$ does give us correctly the result that

$$\int_0^{-\infty} h_1(u)\mathrm{d}u = \int_0^{-\infty} \frac{\mathrm{e}^{u}}{1+u\mathrm{e}^{2u}}\mathrm{d}u = -\overset{\vee}{h}{}_1'(-1) \approx -1.130098101522 \qquad (81)$$

albeit that this integral on $[0, -\infty)$ is not the one which we set out to calculate.

The possibility that our fundamental Taylor-series-to-the-left integration result could perhaps be generalised to a wider class of functions, but applying instead on a suitable contour from 0 to ∞ (not necessarily just $[0, \infty)$) is one we find intriguing. We think it is worthy of further consideration, along with its relationship to the question we have considered here - namely of how to make the inference of TLA-coefficient functions more robust.

(ii) The integral for $h_2(v)$ which we were able to successfully tackle arose from a change of variables regarding $\int_0^1 \frac{1}{1+z^2\ln z}\mathrm{d}z$, while the problematic integral for $h_1(u)$ arose from $\int_1^\infty \frac{1}{1+z^2\ln z}\mathrm{d}z$.

This suggests that in our first example concerning the function $g(z) = \frac{1}{1+z^2\ln z}$, the problems we encountered in pinning down $\overset{\vee}{g}(s)$ arise not really from the positive powers of $z^{2l}(\ln z)^l$ in the Taylor series for $g(z)$ near 0, but rather from the terms $\frac{z^{-2l}}{(\ln z)^l}$ which occur in the power series for $g(z)$ as $z \to \infty$.

Unlike the positive powers, these are not clean generalized eigenfunctions of the Césaro operator, P, and although it is highly speculative to suggest it, this may have some significance in creating the difficulties which arise.

7 Final stray observations and speculations

While in speculative mood, we now conclude both this paper and the series to which it belongs, with some final stray thoughts on Taylor-series-to-the-left methods and where it might be profitable to head next in investigating them - up the Grand Trunk road, further out along the branch line to Marwar junction and beyond, or perhaps up into the hills and mountains of Kafiristan.

Wild speculations and observations: (i) As we just noted , there are functions which are not integrable on $[0,\infty)$ - either classically, or even in a generalised Césaro sense - such as $f(z) = \mathrm{e}^{kz}$ or $f(z) = z^m\mathrm{e}^{kz}$ $(Re(k) > 0)$, to which we can still assign a TLA-coefficient function(albeit perhaps not a canonical one).

For these, we can still calculate $\overset{\vee}{f}'(-1)$ after confirming that $\overset{\vee}{f}(-1) = 0$, and we noted that in these examples, the value of $-\overset{\vee}{f}'(-1)$ can in fact be verified as giving $\int_0^{-\infty} f(z)\mathrm{d}z$. As a result, we speculated that perhaps our whole Taylor-series-to-the-left methodology, and in particular its relationship to generalised Césaro Mellin transforms, could be expanded by relaxing the requirement to focus on $[0,\infty)$ as our standard domain of integration; and instead allowing consideration of alternative contours of integration from 0 to ∞ in the complex plane.

An alternative (or perhaps complementary) possibility is that such examples may be rendered meaningful by extension not via Mellin transforms and the generalised Césaro scheme naturally associated with them, but rather by an alternative generalised convergence scheme (see [I]-[III]) and its naturally associated transform operator. For example, for the sorts of functions just listed, some sort of generalised Borel convergence scheme based on exponential, rather than power, functions may be applicable and may allow us, in conjunction with an allied (presumably Fourier- or Laplace-like) transform, to define analogues of TLA-coefficient functions and derive results analogous to those we have developed in this paper and in [XI]-[XIV].

(ii) As regards Taylor-series-to-the-left methodology based on generalised Césaro convergence and Mellin transforms, however, note that there are not merely many challenges left to investigate, as discussed in section 6. There are also clear boundaries which are insurmountable without some sort of extension along the lines mentioned in (i).

For example, a function like $f(x) = \mathrm{e}^{-x-\frac{1}{x}}$ is Schwartzian both as $x \to \infty$ and as $x \to 0^+$. As such, while it has a well-defined Mellin transform, $-\mathcal{M}[f](-s)$, and therefore a well-defined TLA-coefficient function, $\overset{\vee}{f}(s)$, this TLA-coefficient function has $\overset{\vee}{f}(m) = 0$ for all $m \in \mathbb{Z}$. It is therefore seemingly impossible to get any sort of foothold in trying to use Taylor-series-to-the-left methodology to calculate $\int_0^\infty f(x)\mathrm{d}x$.

(iii) A further natural area for exploration regarding Taylor-series-to-the-left methodology is whether it can be extended from functions of a single variable to higher dimensions and, if so, how.

As regards the integration aspects of the theory, for classically integrable functions of many variables some progress may be achievable simply by adopting polar coordinates and then using results developed so far on the integral component of the form $\int_0^\infty \cdot \, \mathrm{d}r$ (after separating out dependence in r from dependence in the angular variables parametrising the unit sphere).

More generally, however, it is interesting to contemplate how generalised Césaro convergence might be extended from one to higher dimensions; what role the Mellin transform or a suitable analogue of it might play; and how the results we have developed regarding integrals, local and global asymptotics and so forth, might generalise. At the same time, however, we note that the extension to *complex s* has been critical in the one-dimensional theory developed so far - how this connection to complex variables would play out in any such move to higher dimensions is extremely unclear.

(iv) Returning to the 1-d case, another area for speculation concerns the role of power series - near 0, near ∞ or as in, for example, the von Mangoldt formula - in what has been developed.

Such power series are essentially expansions in eigenfunctions of the generalised Césaro operator, P. Thus it is natural to wonder whether there might be value in extending such series to consider expressions which also include *generalised* eigenfunctions of P, namely functions of the form $x^\nu (\ln x)^m$ where $\nu \in \mathbb{C}$ and $m \in \mathbb{Z}_{\geq 0}$; that is, to consider expansions of the form $\sum_{\nu,m} a_{\nu,m} x^\nu (\ln x)^m$.

Note, however, that at this stage this is really nothing more than the strayest of stray thoughts - no progress has been made in this direction either in theory or potential application, despite its seeming naturalness.

(v) Staying with stray observations, the hybrid application of Taylor-series-to-the-left methodology often consists of finding natural methods for going from the functions $f_0(m)$ and $f_\infty(m)$ on $m \in \mathbb{Z}$ - which encode the coefficients of the power series expansions for f near 0 and as $z \to \infty$ - to the general form of $\overset{\vee}{f}(s)$ at arbitrary $s \in \mathbb{C}$, based on knowing that at $s = m \in \mathbb{Z}$ we have

$$\overset{\vee}{f}(m) = f_0(m) - f_\infty(m) \quad .$$

The fact that f_0 and f_∞ are only canonically well-defined for $m \in \mathbb{Z}$ is at the heart of what makes this challenging.

Given this, it is worth noting that, in theory at least, f_0 and f_∞ can actually be defined for arbitrary $s \in \mathbb{C}$, as follows. Fix $a \in (0, \infty)$ arbitrary and define $f_0(s)$ as

$$f_0(s) = \frac{\sin(2\pi s)}{2\pi} \cdot \left\{ -\int_0^a f(z) z^{-s-1} \mathrm{d}z \right\} \tag{82}$$

where the integral is defined as its generalised Césaro value; and likewise define $f_\infty(s)$ as

$$f_\infty(s) = \frac{\sin(2\pi s)}{2\pi} \cdot \left\{ +\int_a^\infty f(z) z^{-s-1} \mathrm{d}z \right\} \quad . \tag{83}$$

Then it is trivial to see that $\overset{\vee}{f}(s) = f_0(s) - f_\infty(s)$ in general for arbitrary $s \in \mathbb{C}$.[13] And an elementary Césaro calculation, taking $\lim_{\epsilon \to 0}$ in the relevant integrals for $s = m + \epsilon$, shows that, independent of what value of a we have chosen, f_0 and f_∞ defined in this way do continue to correctly pick off the coefficients of z^m in the power series for $f(z)$ near 0 and as $z \to \infty$ respectively.

The fact that these results hold irrespective of our choice of a in $(0, \infty)$ shows that we have one parameter of gauge-freedom in how to define f_0 and f_∞ while still satisfying the required conditions prescribing their values at $m \in \mathbb{Z}$.

In practice, these definitions are not ones we have used, since our aim has been to use Taylor-series-to-the-left methods and generalised Césaro techniques to get at the TLA-coefficient function, $\overset{\vee}{f}(s)$, without having to directly calculate either the Mellin transform of f or these two components of it. But it is nice to note that the transition from f_0, f_∞ and $m \in \mathbb{Z}$ to $\overset{\vee}{f}$ and $s \in \mathbb{C}$, which is central to so many of the Taylor-series-to-the-left computations we have undertaken, is a natural, not an artificial, one.

(vi) As a final point, note that while we have focussed on integration results in recent sections and in the earlier comments in this section, we believe that there is large scope for using Taylor-series-to-the-left methods to understand existing results or to derive new ones regarding number theory and the zeta function $\zeta(s)$. We conclude with one further example.

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[13] since the integrals $\int_0^a$ and $\int_a^\infty$ in equations 82 and 83 just split up the Mellin transform $-\mathcal{M}[f](-s)$ underlying our canonical definition of $\overset{\vee}{f}(s)$ into two pieces

In his discussion of the general distribution of zeros of ζ in [2, section 9.8], Titchmarsh proves the result of Ramanujan that for any $a, b \in \mathbb{R}_{>0}$ satisfying $ab = \pi$, we have

$$\sqrt{a} \cdot \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \mathrm{e}^{-\frac{a^2}{n^2}} - \sqrt{b} \cdot \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \mathrm{e}^{-\frac{b^2}{n^2}} = -\frac{1}{2\sqrt{b}} \cdot \sum_{\rho \in NT} b^{\rho} \cdot \frac{\Gamma(\frac{1}{2} - \frac{1}{2}\rho)}{\zeta'(\rho)} . \quad (84)$$

But how Ramanujan came to be considering the expression on the LHS in this equation, and why the subtraction of its two components is both natural and necessary, is left unexplained. One explanation using Césaro arrays and Taylor-series-to-the-left methods to cast light on this can be sketched briefly as follows.

Recall first that in [IV]-[VI] we have considered at length the function[14] $H(a) := \frac{1}{2} + \sum_{n=1}^{\infty} \mathrm{e}^{-\pi n^2 a^2}$. Thinking in terms of Césaro arrays, it is a small step to consider instead summands involving $\mathrm{e}^{-\frac{a^2}{n^2}}$ so that the sums at each "height" in the Césaro array for the Taylor series around 0 are classically convergent. Then, in line with our ruminations towards the end of section 2.4, it is likewise a small step to introduce an overall oscillatory factor involving the Möbius function. If we choose this factor as $\frac{\mu(n)}{n}$ then the sums at each height will still end up involving ζ-values, but they will be at odd rather than even positive integer arguments (and by the functional equation for ζ these will in turn be related to the behaviour of ζ near its trivial zeros at the negative even integers). Finally, incorporating an overall factor which is a fixed power of a outside the sum would not affect any Césaro analysis, and we take this factor as $\sqrt{a}$.[15]

It is thus natural to consider the expression $\sqrt{a} \cdot \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \mathrm{e}^{-\frac{a^2}{n^2}}$, and if we introduce $u := \sqrt{a}$ so as to avoid complexities with branch cuts, this is the function, which we shall call $f(u)$, given by

$$f(u) := u \cdot \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \mathrm{e}^{-\frac{u^4}{n^2}} \quad . \quad (85)$$

This is the first term on the LHS in equation 84. The second term consists of the function

$$g(u) := f(\frac{\sqrt{\pi}}{u}) = \frac{\sqrt{\pi}}{u} \cdot \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \mathrm{e}^{-\frac{\pi^2}{n^2 u^4}} \quad . \quad (86)$$

Now since $\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = \frac{1}{\zeta(1)} = 0$, and since $\lim_{n \to \infty} \mathrm{e}^{-\frac{u^4}{n^2}} = 1$ for any u, it is easy to see that f is neither singular at 0 nor as $u \to \infty$, and the same is true for g.

Considering f first and expanding $\mathrm{e}^{-\frac{u^4}{n^2}}$ as $\sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \frac{1}{n^{2j}} u^{4j}$, the component-1 sums in our Césaro array converge classically to $\frac{(-1)^j}{j!} \frac{1}{\zeta(2j+1)}$ at each height $4j + 1$ and there are no component-2 pieces to include. The Taylor series for f

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[14] which plays a central role in the theory of ζ, going all the way back to Riemann

[15] The reason for this choice emerges in the course of our computation.

near 0 is thus

$$f(u) = \sum_{j=1}^{\infty} \frac{(-1)^j}{j!} \frac{1}{\zeta(2j+1)} u^{4j+1} = -\frac{1}{\zeta(3)} u^5 + \frac{1}{2!} \frac{1}{\zeta(5)} u^9 - \ldots \tag{87}$$

and we have

$$\overset{\vee}{f}(m) = \frac{1}{8} \cdot \frac{\sin(2\pi(m-1))}{\sin(\pi \frac{(m-1)}{4})} \cdot \frac{1}{(\frac{(m-1)}{4})!} \cdot \frac{1}{\zeta(\frac{(m+1)}{2})} \quad \text{for} \quad m \in \mathbb{Z}_{\geq 0} \ . \tag{88}$$

Now, extending to the left, we see that (a) if $m = -4j-3$ then $\overset{\vee}{f}(m) = 0$ since $(\frac{(m-1)}{4})!$ becomes singular and (b) if $m = -4j$ or $m = -4j-2$ is even, then $\overset{\vee}{f}(m) = 0$ because $\sin(2\pi(m-1)) = 0$.

As for the case $m = -4j-1$, we have $\overset{\vee}{f}(-1) = 0$ since $\sin(-4\pi) = 0$ but $\zeta(0) \neq 0$; but for $j \geq 1$ the zeros in $\sin(-4\pi(2j+1))$ and $\zeta(-2j)$ "cancel" each other in the usual way and leave a finite value. An elementary L'Hopital's calculation shows that this is given by

$$\overset{\vee}{f}(-4j-1) = \frac{(-1)^{j+1}}{(-j-\frac{1}{2})!} \cdot \frac{\pi}{2} \cdot \frac{1}{\zeta'(-2j)} \quad . \tag{89}$$

Now the functional equation for ζ says that $\zeta(s) = 2^s \pi^{s-1} \sin(\frac{\pi}{2}s)\Gamma(1-s)\zeta(1-s)$ and it follows on differentiating and setting $s = -2j$ that

$$\zeta'(-2j) = 2^{-2j-1} \cdot \pi^{-2j} \cdot (-1)^j \cdot (2j)! \cdot \zeta(2j+1) \quad .$$

Putting this into equation 89 and recalling from the functional equation for Γ that $\frac{1}{\Gamma(-j+\frac{1}{2})\Gamma(j+\frac{1}{2})} = \frac{\sin(\pi(j+\frac{1}{2}))}{\pi} = \frac{(-1)^j}{\pi}$, it follows that

$$\overset{\vee}{f}(-4j-1) = \frac{(-1)^{j+1} \cdot 2^{2j} \cdot \pi^{2j} \cdot (j-\frac{1}{2})!}{(2j)! \cdot \zeta(2j+1)}$$

and on noting that $(j-\frac{1}{2})! = (j-\frac{1}{2})(j-\frac{3}{2})\cdots(\frac{1}{2})(-\frac{1}{2})! = \frac{(2j-1)(2j-3)\cdots 1}{2^j} \cdot \sqrt{\pi}$, this simplifies to

$$\overset{\vee}{f}(-4j-1) = \frac{(-1)^{j+1} \cdot \pi^{2j+\frac{1}{2}}}{(j)! \cdot \zeta(2j+1)} \quad . \tag{90}$$

Since f is non-singular near 0, the resulting power series in negative powers of u, namely

$$\sum_{j=1}^{\infty} \frac{(-1)^j \cdot \pi^{2j+\frac{1}{2}}}{(j)! \cdot \zeta(2j+1)} \cdot \frac{1}{u^{4j+1}} \tag{91}$$

arises entirely from convergent, decaying behaviour of $f(u)$ as $u \to \infty$. In other words, we have

$$f_0(m) = \begin{cases} \frac{(-1)^j}{(j)! \cdot \zeta(2j+1)} \ , & m = 4j+1 \text{ and } j \geq 1 \\ \\ 0 \ , & else \end{cases} \tag{92}$$

and

$$f_\infty(m) = \begin{cases} \frac{(-1)^j \cdot \pi^{2j+\frac{1}{2}}}{(j)! \cdot \zeta(2j+1)} \ , & m = -4j-1 \text{ and } j \geq 1 \\ \\ 0 \ , & else \ . \end{cases} \tag{93}$$

But then, noting that $g(u) = f(\frac{\sqrt{\pi}}{u})$ and that g is also non-singular both as $u \to 0^+$ and as $u \to \infty$, we can see why it was necessary for Ramanujan to consider the difference $f(u) - g(u)$ as the expression on the LHS in equation 84 rather than $f(u)$ or $g(u)$ alone, or any other combination of them.

This is because, as $u \to 0^+$ we have $\frac{\sqrt{\pi}}{u} \to \infty$ and thus, from equation 91

$$g(u) \approx \sum_{j=1}^{\infty} \frac{(-1)^j \cdot \pi^{2j+\frac{1}{2}}}{(j)! \cdot \zeta(2j+1)} \cdot \frac{1}{(\frac{\sqrt{\pi}}{u})^{4j+1}} = \sum_{j=1}^{\infty} \frac{(-1)^j}{(j)! \cdot \zeta(2j+1)} \cdot u^{4j+1} \tag{94}$$

so that $g_0(m) = f_0(m)$ for all $m \in \mathbb{Z}$; while for $u \to \infty$ we have $\frac{\sqrt{\pi}}{u} \to 0^+$ and therefore, from equation 87

$$g(u) \approx \sum_{j=1}^{\infty} \frac{(-1)^j}{(j)! \cdot \zeta(2j+1)} \cdot (\frac{\sqrt{\pi}}{u})^{4j+1} = \sum_{j=1}^{\infty} \frac{(-1)^j \cdot \pi^{2j+\frac{1}{2}}}{(j)! \cdot \zeta(2j+1)} \cdot \frac{1}{u^{4j+1}} \tag{95}$$

so that $g_\infty(m) = f_\infty(m)$ for all $m \in \mathbb{Z}$.

It follows that, while $f(u)$ and $g(u)$ are *not* the same function, they have identical power series expansions both near 0 and as $u \to \infty$. Thus their difference has both such power series expressions being identically zero. In other words, their difference is Schwartzian both as $u \to 0^+$ and as $u \to \infty$, i.e.

$$f(u) - g(u) \in \mathcal{S}_0(u) \qquad \text{and} \qquad f(u) - g(u) \in \mathcal{S}_\infty(u) \quad . \tag{96}$$

This is necessary in order for Ramanujan's formula 84 to have any chance of holding, since the RHS in equation 84 only contains powers $u^{-2\rho+1}$, $\rho \in NT$, and therefore could not entail integer powers of u as $u \to 0^+$ or $u \to \infty$.

Comments: **(i)** The above only shows, of course, why it was *necessary* for Ramanujan to take the difference $f(u) - g(u)$ on the LHS in this formula; it is not *sufficient* to show why this remarkable formula holds. That proof involves contour integration and is given in [2, section 9.8].

We nonetheless believe that the explanation and argument given here provide insight into the structure of the formula. And, as noted, we believe there is considerable scope for applying such Césaro array and Taylor-series-to-the-left reasoning - which we find reminiscent of dimensional analysis in physics - more widely in the areas of analytic number theory and the analysis of ζ.

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